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Systems of Parabolic Equations with Continuous and Discrete Delays

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In this paper we investigate the global existence and the dynamics of a coupled system of nonlinear parabolic equations where the nonlinear "reaction function" may depend on both continuous (infinite or finite) and discrete delays. It is shown that if the reaction function is locally Lipschitz continuous and the system possesses a pair of coupled upper and lower solutions then there exists a unique global solution to the system without any quasimonotone condition on the reaction function. For systems with mixed quasimonotone reaction functions we use the monotone method to establish more dynamic property of the parabolic system in terms of the quasisolutions of the corresponding elliptic system. This approach yields a (global) attractor of the parabolic system, and under some additional conditions this attractor leads to the existence and asymptotic stability of a solution of the elliptic system. Application to three model problems in population dynamics and chemical reactions is given. © 1997 Academic Press

1. INTRODUCTION

In recent years considerable attention has been given to parabolic partial differential equations with delays, especially in relation to reaction-diffusion systems where the reaction function depends on the unknown function with time delays. However, most of the discussions are either for scalar parabolic equations or for coupled systems with discrete delays (cf. [5–10, 12–16, 18]). In this paper we consider a coupled system of parabolic equations where the reaction function may depend on both continuous (infinite or finite) and discrete delays. The system of equations

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under consideration is given in the form

$$
\begin{aligned}
\partial u_i/\partial t - L_i u_i &= f_i(t, x, \mathbf{u}, J * \mathbf{u}) && (t > 0, x \in \Omega) \\
B_i u_i &= h_i(t, x) && (t > 0, x \in \partial\Omega) \quad (i = 1, \dots, N) \\
u_i(t, x) &= \eta_i(t, x) && (t \in I_i, x \in \Omega),
\end{aligned}
\tag{1.1}
$$

where $\mathbf{u} \equiv (u_1, \dots, u_N)$, $\mathbf{J} * \mathbf{u} = (J_1 * u_1, \dots, J_N * u_N)$, Ω is a bounded domain in $\mathbb{R}^p$ $(p = 1, 2, \dots)$ with boundary $\partial\Omega$, and for each $i = 1, \dots, N$, L_i and B_i are the elliptic and boundary operators given by

$$
\begin{aligned}
L_i u_i &= \sum_{j,k=1}^{p} a_{jk}^{(i)}(t, x)\, \partial^2 u_i/\partial x_j\, \partial x_k + \sum_{j=1}^{p} b_j^{(i)}(t, x)\, \partial u_i/\partial x_j \\
B_i u_i &= \alpha_i \partial u_i/\partial \nu + \beta_i(t, x) u_i \qquad (i = 1, \dots, N).
\end{aligned}
\tag{1.2}
$$

In (1.2), $\partial/\partial\nu$ denotes the outward normal derivative on $\partial\Omega$, and the boundary coefficients α_i and β_i are given either by $\alpha_i = 0$, $\beta_i = 1$ (Dirichlet condition) or by $\alpha_i = 1$, $\beta_i \geq 0$ (Neumann or Robin condition). The interval I_i and the components $J_i * u_i$ in (1.1) are given, respectively, by

$$
I_i = \begin{cases} (-\infty, 0] & \text{when } i = 1, \dots, n_0 \\ [-r_i, 0] & \text{when } i = n_0 + 1, \dots, N, \end{cases}
\tag{1.3}
$$

and

$$
J_i * u_i = \begin{cases} \displaystyle\int_{-\infty}^{t} J_i(t - s, x) u_i(s, x)\, ds & \text{when } i = 1, \dots, n_0 \\ u_i(t - r_i, x) & \text{when } i = n_0 + 1, \dots, N. \end{cases}
\tag{1.4}
$$

The integral term in (1.4) represents an infinite continuous delay while $u_i(t - r_i, x)$ denotes a discrete delay. If for some or all i with $1 \leq i \leq n_0$ the continuous delay is given by the finite integral

$$
J_i * u_i = \int_{-r_i}^{t} J_i(t - s, x) u_i(s, x)\, ds,
\tag{1.5}
$$

where $0 < r_i < \infty$, then the semi-infinite interval I_i in (1.3) is replaced by $[-r_i, 0]$. The above consideration implies that system (1.1) may consist of a combination of finite and infinite continuous delays as well as discrete delays. In the special case of $n_0 = N$ or n_0 is zero the delays in the system are either of continuous type or of discrete type, respectively. When n_0 is not zero we assume that for each $i = 1, \dots, n_0$, $J_i(t, x)$ is piecewise continuous in t, Holder continuous in x (uniformly in t), and possesses the property

$$J_i(t,x) \geq 0 \text{ for } t \geq 0 \qquad \text{and} \qquad \int_0^{\infty} J_i(t,x)\,dt = 1\ (x \in \Omega). \quad (1.6)$$

For finite continuous delay the above condition is replaced by

$$J_i(t,x) \geq 0 \text{ for } 0 \leq t \leq r_i, \qquad J_i(t,x) \equiv 0 \text{ for } t > r_i,$$

and

$$\int_0^{r_i} J_i(t,x)\,dt = 1 \qquad (x \in \Omega). \quad (1.7)$$

Unless otherwise stated we use the integral form $J_i * u_i$ in (1.4) for either infinite or finite continuous delay.

In the system (1.1) some of the parabolic equations may be replaced by ordinary differential equations. Since the treatment for coupled parabolic-ordinary systems is similar to that for parabolic systems with $L_i = 0$ (and without the corresponding boundary condition) we only consider the parabolic system (1.1) with the understanding that L_i may be identically zero for some values of i (see [10, 12] for some detailed discussions).

Parabolic systems in the form (1.1) have been investigated by many investigators, and most of the discussions are devoted to the existence and asymptotic behavior of the solution (cf. [7–9, 12–16, 18]). In earlier works the system is usually formulated as an evolution equation of functional type, and the existence-dynamic problem is investigated by semi-group approach (e.g., see [8, 13–16]). Recently, the method of upper and lower solutions and its associated monotone iterations have been used to study the existence and dynamics of (1.1) for the case of discrete delays (cf. [5, 10, 12]). An advantage of the monotone method is that it leads to a computational algorithm for the computation of numerical solutions (e.g., see [5, 6]). However, in the iteration process by the monotone method it is required that the reaction function $\mathbf{f} \equiv (f_1, \ldots, f_N)$ in (1.1) possesses a mixed quasimonotone property in the sector between upper and lower solutions.

The purpose of this paper is twofold: (1) to show the existence and uniqueness of a solution for the more general delayed system (1.1) without any quasimonotone condition on the nonlinear function $(f_1, \ldots, f_N)$, and (2) to investigate the dynamic property of (1.1) by establishing a global attractor in relation to the quasisolutions of the corresponding elliptic system. In some cases this attractor leads to the asymptotic stability of a solution of the elliptic system. The existence-uniqueness result in item (1) is given in Section 2 while the dynamic problem is discussed in Section 3. In Section 4, we give an application of these results to three model problems arising from population dynamics and chemical reactions.

2. EXISTENCE THEOREMS FOR NONQUASIMONOTONE FUNCTIONS

Let $T > 0$ be an arbitrary finite number, and let $\overline{\Omega} = \Omega \cup \partial\Omega$. Set

$$\begin{aligned}
&D_T = (0, T] \times \Omega, \quad S_T = (0, T] \times \partial\Omega, \quad \overline{D}_T = [0, T] \times \overline{\Omega},\\
&Q_0^{(i)} = (-\infty, 0] \times \Omega, \quad Q_T^{(i)} = (-\infty, T] \times \Omega\\
&\overline{Q}_T^{(i)} = (-\infty, T] \times \overline{\Omega}, \text{ for } i = 1, \dots, n_0,\\
&Q_0^{(i)} = [-r_i, 0] \times \Omega, \quad Q_T^{(i)} = [-r_i, T] \times \Omega,\\
&\overline{Q}_T^{(i)} = [-r_i, T] \times \overline{\Omega}, \text{ for } i = n_0 + 1, \dots, N,\\
&Q_0 = Q_0^{(1)} \times \cdots \times Q_0^{(N)}, \quad Q_T = Q_T^{(1)} \times \cdots \times Q_T^{(N)},\\
&\overline{Q}_T = \overline{Q}_T^{(1)} \times \cdots \times \overline{Q}_T^{(N)}.
\end{aligned}$$

Denote by $C^\alpha(D_T)$ the space of Holder continuous functions in D_T with exponent $\alpha \in (0, 1)$, and by $C^{1,2}(D_T)$ the class of functions which are once continuously differentiable in t and twice continuously differentiable in x. The space of continuous functions in $\overline{D}_T$ is denoted by $C(\overline{D}_T)$. For vector-valued functions we use the product spaces

$$\begin{aligned}
\mathscr{C}(\overline{D}_T) &\equiv C(\overline{D}_T) \times \cdots \times C(\overline{D}_T),\\
\mathscr{C}^\alpha(D_T) &\equiv C^\alpha(D_T) \times \cdots \times (D_T).
\end{aligned}$$

Similar notations are used for other function spaces and other domains. We assume that for each $i = 1, \dots, N$, the coefficients of L_i and the first partial derivatives of $(a_{j,k}^{(i)})$ are in $C^\alpha(\overline{D}_T)$, the boundary coefficient β_i is in $C^\alpha(S_T)$, and $\partial\Omega$ is of class $C^{1+\alpha}$. The function $f_i(t, x, \mathbf{u}, \mathbf{v})$ is assumed Holder continuous in $(t, x) \in \overline{D}_T$ and locally Lipschitz continuous in $(\mathbf{u}, \mathbf{v})$. The boundary and initial data h_i, η_i are assumed Holder continuous on S_T and $Q_0^{(i)}$, respectively, and η_i satisfies the compatibility condition $\eta_i(0, x) = h_i(0, x)$ when $\alpha_i = 0$, $\beta_i = 1$. For infinite delays we also assume that $\eta_i(\cdot, t) \in L^1(-\infty, 0)$ (that is, $\int_{-\infty}^0 |\eta_i(t, x)|\, dt < \infty$). The above general assumptions are used to ensure the existence of a classical solution to (1.1). Our approach to the problem is by the method of upper and lower solutions which are defined as follows:

DEFINITION 2.1. Two functions $\tilde{\mathbf{u}} \equiv (\tilde{u}_1, \dots, \tilde{u}_N)$, $\hat{\mathbf{u}} \equiv (\hat{u}_1, \dots, \hat{u}_N)$ in $\mathscr{C}^\alpha(\overline{Q}_T) \cap \mathscr{C}^{1,2}(D_T)$ are called coupled upper and lower solutions of (1.1) if $\tilde{\mathbf{u}} \geq \hat{\mathbf{u}}$ in $\overline{Q}_T$ and if

$$\begin{aligned}
&\partial \tilde{u}_i/\partial t - L_i \tilde{u}_i \geq f_i(t, x, \mathbf{v}, \mathbf{w}) \quad \text{for all } \mathbf{v}, \mathbf{w} \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle \text{ with } v_i = \tilde{u}_i \\
&\partial \hat{u}_i/\partial t - L_i \hat{u}_i \leq f_i(t, x, \mathbf{v}, \mathbf{w}) \quad \text{for all } \mathbf{v}, \mathbf{w} \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle \text{ with } v_i = \hat{v}_i \\
&\qquad B_i \tilde{u}_i \geq h_i(t, x) \geq B_i \hat{u}_i \qquad \text{on } S_T \\
&\qquad \tilde{u}_i(t, x) \geq \eta_i(t, x) \geq \hat{u}_i(t, x) \quad \text{in } Q_0^{(i)} \ (i = 1, \dots, N).
\end{aligned} \tag{2.1}$$

In the above definition, inequalities between two vectors are in the componentwise sense and

$$\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle \equiv \left\{ \mathbf{u} \in \mathscr{C}\left(\overline{Q}_T\right); \hat{\mathbf{u}} \leq \mathbf{u} \leq \tilde{\mathbf{u}} \right\}. \tag{2.2}$$

It is easy to see that the above definition of upper and lower solutions coincides with the definition in [10, 12] when the reaction function $\mathbf{f} \equiv (f_1, \dots, f_N)$ is mixed quasimonotone. Throughout this section we assume that a pair of coupled upper and lower solutions exist, and for each $i = 1, \dots, N$, f_i satisfies the Lipschitz condition

$$\left| f_i(t, x, \mathbf{u}, \mathbf{w} - f_i(t, x, \mathbf{u}', \mathbf{w}')) \right| \leq K_i(|\mathbf{u} - \mathbf{u}'| + |\mathbf{w} - \mathbf{w}'|)$$
$$(\mathbf{u}, \mathbf{u}', \mathbf{w}, \mathbf{w}' \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle) \tag{2.3}$$

for some constant $K_i > 0$, where $|\mathbf{u}| \equiv |u_1| + \cdots + |u_N|$ for any $\mathbf{u} = (u_1, \dots, u_N)$ in $\mathbb{R}^N$.

For each $i = 1, \dots, N$, we define a truncated function $\hat{f}_i$ such that

$$\hat{f}_i(t, x, \mathbf{u}, \mathbf{w}) = f_i(t, x, \mathbf{u}, \mathbf{w}) \quad \text{when } u, \mathbf{w} \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$$

and $\hat{f}_i$ satisfies the global Lipschitz condition

$$\left| \hat{f}_i(t, x, \mathbf{u}, \mathbf{w}) - \hat{f}_i(t, x, \mathbf{u}', \mathbf{w}') \right| \leq K_i(|\mathbf{u} - \mathbf{u}'| + |\mathbf{w} - \mathbf{w}'|)$$
$$\text{for all } \mathbf{u}, \mathbf{u}', \mathbf{w}, \mathbf{w}' \in \mathbb{R}^N. \tag{2.4}$$

This can be done by replacing the component (u_j, w_l) of $(\mathbf{u}, \mathbf{w})$ in $f_i(\cdot, \mathbf{u}, \mathbf{w})$ by the corresponding component $(\tilde{u}_j, \tilde{w}_l)$ when $(u_j, w_l) > (\tilde{u}_j, \tilde{w}_l)$, and by $(\hat{u}_j, \hat{w}_l)$ when $(u_j, w_l) < (\hat{u}_j, \hat{w}_l)$. With this modification the pair $\tilde{\mathbf{u}}, \hat{\mathbf{u}}$ satisfy the relation

$$\begin{aligned}
&\partial \tilde{u}_i/\partial t - L_i \tilde{u}_i \geq \hat{f}_i(t, x, \mathbf{v}, \mathbf{w}) \quad \text{for all } \mathbf{v}, \mathbf{w} \in \mathscr{C}\left(\overline{D}_T\right) \text{ with } v_i = \tilde{u}_i \\
&\partial \hat{u}_i/\partial t - L_i \hat{u}_i \leq \hat{f}_i(t, x, \mathbf{v}, \mathbf{w}) \quad \text{for all } \mathbf{v}, \mathbf{w} \in \mathscr{C}\left(\overline{D}_T\right) \text{ with } v_i = \hat{u}_i \\
&\qquad (i = 1, \dots, N).
\end{aligned} \tag{2.5}$$

Let $\mathbf{u}^i$ denote the vector $\mathbf{u}$ where the ith component of $\mathbf{u}$ is replaced by u_i'. Then the global Lipschitz condition (2.4) implies that

$$K_i u_i + \hat{f}_i(\cdot, \mathbf{u}, \mathbf{w}) \geq K_i u_i' + \hat{f}_i(\cdot, \mathbf{u}^i, \mathbf{w})$$
$$\text{for all } \mathbf{u}, \mathbf{w} \in \mathbb{R}^N \text{ whenever } u_i \geq u_i'. \quad (2.6)$$

Our first goal is to show that the truncated problem (1.1), where f_i is replaced by $\hat{f}_i$, has a unique global solution. To this goal we use an initial iteration $\mathbf{u}^{(0)} \in \mathscr{C}^{\alpha}(D_T) \cap \mathscr{C}(\overline{D}_T)$ and construct a sequence $\{\mathbf{u}^{(k)}\} \equiv \{u_1^{(k)}, \ldots, u_N^{(k)}\}$ from the linear iteration process

$$\begin{aligned} \mathscr{L}_i u_i^{(k)} &= K_i u_i^{(k-1)} + \hat{f}_i(t, x, \mathbf{u}^{(k-1)}, \mathbf{J} * \mathbf{u}^{(k-1)}) \qquad \text{in } D_T \\ B_i u_i^{(k)} &= h_i(t, x) \qquad \text{on } S_T \\ u_i^{(k)}(t, x) &= \eta_i(t, x) \qquad \text{in } Q_0^{(i)} \ (i = 1, \ldots, N), \end{aligned} \quad (2.7)$$

where $k = 1, 2, \ldots, K_i$ is the Lipschitz constant in (2.4), and

$$\mathscr{L}_i u_i = \partial u_i / \partial t - L_i u_i + K_i u_i.$$

The above construction is crucial for proving the existence of a unique solution to the original problem (1.1).

To show the convergence of the sequence $\{\mathbf{u}^{(k)}\}$ we make the transformation $\mathbf{w} = e^{-\gamma t}\mathbf{u}$ for some positive constant $\gamma > 0$, and transform the truncated problem (1.1) to the form

$$\begin{aligned} \mathscr{L}_i w_i + \gamma w_i &= f_i^*(t, x, \mathbf{w}, \mathbf{J} * \mathbf{w}) \qquad \text{in } D_T \\ B_i w_i &= h_i^*(t, x) \qquad \text{on } S_T \\ w_i(t, x) &= \eta_i^*(t, x) \qquad \text{in } Q_0^{(i)} \ (i = 1, \ldots, N), \end{aligned} \quad (2.8)$$

where $h_i^* = e^{-\gamma t} h_i$, $\eta_i^* = e^{-\gamma t}\eta_i$, and

$$f_i^*(t, x, \mathbf{w}, \mathbf{J} * \mathbf{w}) = K_i w_i + e^{-\gamma t}\hat{f}_i\big(t, x, e^{\gamma t}\mathbf{w}, \mathbf{J} * (e^{\gamma t}\mathbf{w})\big). \quad (2.9)$$

Similarly the transformation $\mathbf{w}^{(k)} = e^{-\gamma t}\mathbf{u}^{(k)}$ transforms the iteration process (2.7) to the form

$$\begin{aligned} \mathscr{L}_i w_i^{(k)} + \gamma w_i^{(k)} &= f_i^*(t, x, \mathbf{w}^{(k-1)}, \mathbf{J} * \mathbf{w}^{(k-1)}) \qquad \text{in } D_T \\ B_i w_i^{(k)} &= h_i^*(t, x) \qquad \text{on } S_T \\ w_i^{(k)} &= \eta_i^*(t, x) \qquad \text{in } Q_0^{(i)} \ (i = 1, \ldots, N), \end{aligned} \quad (2.10)$$

where $\mathbf{w}^{(k)} \equiv (w_1^{(k)}, \ldots, w_N^{(k)})$. By the integral representation for linear parabolic boundary-value problems, the sequences $\{\mathbf{w}^{(k)}\}$ may be expressed by

$$w_i^{(k)}(t, x) = \int_0^t d\tau \int_\Omega \Gamma_i(t, x; \tau, \xi)\big(f_i^*(\mathbf{w}^{(k-1)}, \mathbf{J} * \mathbf{w}^{(k-1)})\big)(\tau, \xi)\, d\xi$$
$$+ \int_0^t d\tau \int_{\partial\Omega} \Gamma_i(t, x; \tau, \xi)\, \psi_i^{(k-1)}(\tau, \xi)\, d\xi + I_i^{(0)}(t, x) \quad (2.11)$$

when $\alpha_i = 1$, $\beta_i \geq 0$ (Neumann and Robin condition), and by

$$w_i^{(k)}(t,x) = \int_0^t d\tau \int_\Omega G_i(t,x;\tau,\xi)\big(f_i^*(\mathbf{w}^{(k-1)}, \mathbf{J}*\mathbf{w}^{(k-1)})\big)(\tau,\xi)\,d\xi$$
$$+ \int_0^t d\tau \int_{\partial\Omega} \frac{\partial \Gamma_i}{\partial \nu_\xi}(t,x;\tau,\xi)\psi_i(\tau,\xi)\,d\xi + I_i^{(1)}(t,x) \quad (2.12)$$

when $\alpha_i = 0$, $\beta_i = 1$ (Dirichlet condition), where

$$\big(f_i^*(\mathbf{w}, \mathbf{J}*\mathbf{w})\big)(t,x) \equiv f_i^*(t,x,\mathbf{w}(t,x), (\mathbf{J}*\mathbf{w})(t,x)). \quad (2.13)$$

In the above integral representations, Γ_i and G_i are the respective fundamental solution and Green's function of $\mathscr{L}_i + \gamma$, $\psi_i^{(k)}$ and ψ_i are the respective densities of the single layer and double layer potentials, and $I_i^{(0)}$ and $I_i^{(1)}$ are given by

$$I_i^{(0)}(t,x) = \int_\Omega \Gamma_i(t,x;0,\xi)\eta_i(0,\xi)\,d\xi$$

$$I_i^{(1)}(t,x) = \int_\Omega G_i(t,x;0,\xi)\eta_i(0,\xi)\,d\xi$$

(cf. [3, 9, 12]). We show that the sequence $\{\mathbf{w}^{(k)}\}$ converges in $\mathscr{C}(\overline{D}_T)$ to a unique solution of the corresponding integral equations (2.11)–(2.12). When no confusion arises we use the same norm notation $\|\cdot\|_0$ for the space $\mathscr{C}(\overline{D}_T)$ as well as for $\mathscr{C}(\overline{D}_T)$. The following lemma gives some properties of the function $f_i^*(\mathbf{w}, \mathbf{J}*\mathbf{w})$ in the space $\mathscr{X} \equiv X_1 \times \cdots \times X_N$, where

$$X_i \equiv \left\{ w_i \in \mathscr{C}^\alpha(D_T) \cap C\big(\overline{D}_T\big);\, w_i = \eta_i^* \text{ in } Q_0^{(i)} \right\}, \qquad i = 1,\ldots,N. \quad (2.14)$$

LEMMA 2.1. *Let condition* (2.4) *hold. Then for any* $\mathbf{w}, \mathbf{w}' \in \mathscr{X}$, $f_i^*(\mathbf{w}, \mathbf{J}*\mathbf{w}) \in C^\alpha(D_T) \cap C(\overline{D}_T)$ *and*

$$\left| f_i^*(\mathbf{w}, \mathbf{J}*\mathbf{w}) - f_i(\mathbf{w}', \mathbf{J}*\mathbf{w}') \right| \leq 3K_i \|\mathbf{w} - \mathbf{w}'\|_0 \qquad (i = 1,\ldots,N). \quad (2.15)$$

Proof. To prove $f_i^*(\mathbf{w}, \mathbf{J}*\mathbf{w}) \in C^\alpha(D_T) \cap C(\overline{D}_T)$ it suffices to show the same for $\hat{f}_i(\mathbf{u}, \mathbf{J}*\mathbf{u})$ when $\mathbf{u} \in \mathscr{C}^\alpha(D_T) \cap \mathscr{C}(\overline{D}_T)$ with $u_i = \eta_i$ in $Q_0^{(i)}$. Consider the case $i = 1,\ldots,n_0$ for continuous delay. Without loss of generality we may only treat the case of infinite delay. By (1.4),

$$\begin{aligned}(J_i * u_i)(t,x) &= \int_{-\infty}^{0} J_i(t-s,x)\eta_i(s,x)\,ds + \int_0^t J_i(s,x)u_i(t-s,x)\,ds \\ &\equiv I_1(t,x) + I_2(t,x).\end{aligned}$$

Let $(t,x),(t',x') \in D_T$. By (1.5), $\eta_i \in L^1(-\infty,0)$, and the Holder continuity property of J_i and η_i, we have

$$\begin{aligned}\left|I_i^{(1)}(t,x) - I_i^{(1)}(t',x')\right| &\le \int_{-\infty}^{0} \left|\left(J_i(t-s,x) - J_i(t'-s,x')\right)\eta_i(s,x)\right| ds \\ &\quad + \int_{-\infty}^{0} \left|J_i(t'-s,x')\left(\eta_i(s,x) - \eta_i(s,x')\right)\right| ds \\ &\le c_i\left(|t-t'|^\alpha + |x-x'|^\alpha\right) + c_i\,|x-x'|^\alpha\end{aligned}$$

for some constant c_i. Similarly,

$$\begin{aligned}&\left|I_i^{(2)}(t,x) - I_i^{(2)}(t',x')\right| \\ &\quad\le \int_{t'}^{t} \left|\left(J_i(s,x)u_i(t-s,x)\right)\right| ds \\ &\qquad + \int_0^{t'} \left|J_i(s,x)u_i(t-s,x) - J_i(s,x')u_i(t'-s,x')\right| ds \\ &\quad\le c_i'\,|t-t'| + c_i'\left(|t-t'|^\alpha + |x-x'|^\alpha\right)\end{aligned}$$

for some constant c_i'. This shows that $J_i * u_i$ is Holder continuous in D_T for $i = 1,\dots,n_0$. In view of (1.4), $J_i * u_i$ is Holder continuous in D_T for $i = n_0+1,\dots,N$. It follows from the Holder continuity of $\hat{f}_i(t,x,\mathbf{u},\mathbf{J}*\mathbf{u})$ in (t,x) and the Lipschitz condition (2.4) that $\hat{f}_i(\mathbf{u},\mathbf{J}*\mathbf{u})$ is Holder continuous in D_T. This proves $\hat{f}_i(\mathbf{u},\mathbf{J}*\mathbf{u}) \in C^\alpha(D_T) \cap C(\overline{D}_T)$ for $i = 1,\dots,N$.

To show the relation (2.15) we observe from (2.4) and (2.9) that

$$\begin{aligned}&\left|f_i^*(\mathbf{w},\mathbf{J}*\mathbf{w}) - f_i^*(\mathbf{w}',\mathbf{J}*\mathbf{w}')\right| \\ &\quad\le K_i\,|w_i - w_i'| + e^{-\gamma t}\Big|\hat{f}_i\left(e^{\gamma t}\mathbf{w},\mathbf{J}*(e^{\gamma t}\mathbf{w})\right) \\ &\qquad\qquad - \hat{f}_i\left(e^{\gamma t}\mathbf{w}',\mathbf{J}*(e^{\gamma t}\mathbf{w}')\right)\Big| \qquad (2.16) \\ &\quad\le K_i\,|w_i - w_i'| + K_i e^{-\gamma t}\Big[e^{\gamma t}\,|\mathbf{w}-\mathbf{w}'| \\ &\qquad\qquad + \left|\,\mathbf{J}*\left(e^{\gamma t}(\mathbf{w}-\mathbf{w}')\right)\right|\Big].\end{aligned}$$

Since by (1.4), (1.5), and $\mathbf{w} = \mathbf{w}' = \boldsymbol{\eta}^*$ in Q_0,

$$\begin{aligned}\left|J_i * \left(e^{\gamma t}(w_i - w_i')\right)\right| &\le \int_0^t J_i(t-s,x)e^{\gamma s}\,|w_i(s,x) - w_i'(s,x)|\,ds \\ &\le e^{\gamma t}\,\|w_i - w_i'\|_0\end{aligned}$$

for $i = 1, \dots, n_0$, and

$$\left| J_i * \left(e^{\gamma t}(w_i - w_i')\right)\right| = e^{\gamma t}\left|w_i(t - r_i, x) - w_i'(t - r_i, x)\right|$$
$$\leq e^{\gamma t}\|w - w'\|_0$$

for $i = n_0 + 1, \dots, N$, we see that

$$\left|\mathbf{J} * \left(e^{\gamma t}(\mathbf{w} - \mathbf{w}')\right)\right| \leq e^{\gamma t}\left(\|w_1 - w_1'\|_0 + \cdots + \|w_N - w_N'\|_0\right)$$
$$= e^{\gamma t}\|\mathbf{w} - \mathbf{w}'\|_0.$$

Using the above estimate in (2.16) leads to

$$\left| f_i^*(\mathbf{w}, \mathbf{J} * \mathbf{w}) - f_i^*(\mathbf{w}', \mathbf{J} * \mathbf{w}')\right| \leq K_i\,|w_i - w_i'| + 2K_i\,\|\mathbf{w} - \mathbf{w}'\|_0$$
$$\leq 3K_i\,\|\mathbf{w} - \mathbf{w}'\|_0.$$

This proves the lemma. ∎

Based on the global Lipschitz condition (2.15) we have the following existence result for the truncated problem.

THEOREM 2.1. *Let $\hat{f}_i$ satisfy condition* (2.4). *Then the truncated problem* (1.1) *where f_i is replaced by $\hat{f}_i$ has a unique solution* $\mathbf{u}^*(t, x)$. *Moreover, for any* $\mathbf{u}^{(0)} \in \mathscr{C}^{\alpha}(D_T) \cap \mathscr{C}(\overline{D}_T)$ *with* $\mathbf{u}^{(0)} = \boldsymbol{\eta}$ *in Q_0 the sequence* $\{\mathbf{u}^{(k)}\}$ *obtained from* (2.7) *converges to* $\mathbf{u}^*$ *as $k \to \infty$.*

Proof. We prove the theorem by using the contraction mapping theorem in $\mathscr{C}(\overline{D}_T)$ for the transformed problem (2.8). For each $i = 1, \dots, N$, define operators $A_i\colon D(A_i) \to R(A_i)$ and $F_i^*\colon \mathscr{X} \to C^{\alpha}(D_T) \cap C(\overline{D}_T)$ by

$$\begin{aligned} A_i w_i &\equiv \mathscr{L}_i w_i + \gamma w_i \qquad \left(w_i \in D(A_i)\right) \\ F_i^*(\mathbf{w}) &\equiv f_i^*(\mathbf{w}, \mathbf{J} * \mathbf{w}) \qquad (\mathbf{w} \in \mathscr{X}), \end{aligned} \tag{2.17}$$

where $D(A_i)$ is the domain of A_i given by

$$D(A_i) = \left\{w_i \in C^{1,2}(D_T) \cap C\left(\overline{D}_T\right);\ Bw_i = h_i^* \text{ on } S_T,\ w_i = \boldsymbol{\eta}_i^* \text{ in } Q_0^{(i)}\right\},$$

$R(A_i)$ is the range of A_i, and $f_i^*(\mathbf{w}, \mathbf{J} * \mathbf{w})$ is given by (2.13). It is clear from Lemma 2.1 that F_i^* maps $\mathscr{X}$ into $C^{\alpha}(D_T) \cap C(\overline{D}_T)$. In terms of the operators A_i and F_i^* the iteration process (2.10) may be written as

$$A_i w_i^{(k)} = F_i^*(\mathbf{w}^{(k-1)}) \qquad \left(w_i^{(k)} \in D(A_i)\right)\ i = 1, \dots, N,$$

and in vector form it becomes

$$\mathscr{A}\mathbf{w}^{(k)} = F^*(\mathbf{w}^{(k-1)}) \qquad \left(\mathbf{w}^{(k)} \in D(\mathscr{A})\right), \tag{2.18}$$

where $D(\mathscr{A}) = D(A_1) \times \cdots \times D(A_N)$ and

$$\mathscr{A}\mathbf{w} = (A_1 w_1, \ldots, A_N w_N)$$

$$F^*(\mathbf{w}) = \big(F_1^*(\mathbf{w}), \ldots, F_N^*(\mathbf{w})\big).$$

It is known that the inverse operator $\mathscr{A}^{-1}$ exists and possesses the property

$$\|\mathscr{A}^{-1}\mathbf{w} - \mathscr{A}^{-1}\mathbf{w}'\|_0 \leq (\gamma + \underline{K})^{-1} \|\mathbf{w} - \mathbf{w}'\|_0$$
$$\text{for } \mathbf{w}, \mathbf{w}' \in \mathscr{C}^{\alpha}(D_T) \cap \mathscr{C}\big(\overline{D}_T\big) \tag{2.19}$$

where $\underline{K} \equiv \min\{K_1, \ldots, K_n\}$ (cf. [9]). This implies that eq. (2.18) is equivalent to

$$\mathbf{w}^{(k)} = \mathscr{A}^{-1}F^*(\mathbf{w}^{(k-1)}) \qquad \big(\mathbf{w}^{(k)} \in D(\mathscr{A})\big) \tag{2.20}$$

which may be considered as a compact form for the integral representation (2.11)–(2.12) in the space $\mathscr{C}^{\alpha}(D_T) \cap \mathscr{C}(\overline{D}_T)$. Since by (2.15) there exists a constant K, independent of γ, such that

$$\| F^*(\mathbf{w}) - F^*(\mathbf{w}') \|_0 \leq K \|\mathbf{w} - \mathbf{w}'\|_0 \qquad \text{for } \mathbf{w}, \mathbf{w}' \in \mathscr{X},$$

the relation (2.19) implies that

$$\big\|\mathscr{A}^{-1}F^*(\mathbf{w}) - \mathscr{A}^{-1}F^*(\mathbf{w}')\big\|_0 \leq K(\gamma + \underline{K})^{-1} \|\mathbf{w} - \mathbf{w}'\|_0 \qquad \text{for } \mathbf{w}, \mathbf{w}' \in \mathscr{X}.$$

It follows by choosing $\gamma > K$ that $(\mathscr{A}^{-1}F^*)$ possesses a contraction property in $\mathscr{X}$. This ensures that the sequence $\{\mathbf{w}^{(k)}\}$ converges in $\mathscr{C}(\overline{D}_T)$ to some $\mathbf{w}^* \in \mathscr{C}(\overline{D}_T)$. By the equivalence between (2.20) and (2.11)–(2.12) the sequences $\{w_i^{(k)}\}$ given by (2.11) and (2.12) converge in $C(\overline{D}_T)$ to w_i^*, where w_i^*, $i = 1, \ldots, N$, are the components of $\mathbf{w}^*$. Letting $k \to \infty$ in (2.11) and (2.12) the argument used in [10] shows that $\psi_i^{(k)} \to \psi_i^*$ for some continuous function ψ_i^*, and w_i^* satisfies the integral equation

$$\begin{aligned} w_i^*(t, x) = {} & \int_0^t d\tau \int_{\Omega} \Gamma_i(t, x; \tau, \xi)\big(f_i^*(\mathbf{w}^*, \mathbf{J} * \mathbf{w}^*)(\tau, \xi)\big)\, d\xi \\ & + \int_0^t d\tau \int_{\partial\Omega} \Gamma_i(t, x; \tau, \xi)\, \psi_i^*(\tau, \xi)\, d\xi + I_i^{(0)}(t, x) \end{aligned} \tag{2.21}$$

when $\alpha_i = 1$, $\beta_i \geq 0$, and it satisfies the integral equation

$$\begin{aligned} w_i^*(t, x) = {} & \int_0^t d\tau \int_{\Omega} G_i(t, x; \tau, \xi)\big(f_i^*(\mathbf{w}^*, \mathbf{J} * \mathbf{w}^*)\big)(\tau, \xi)\, d\xi \\ & + \int_0^t d\tau \int_{\partial\Omega} \frac{\partial \Gamma_i}{\partial \nu_{\xi}}(t, x; \tau, \xi)\, \psi_i(\tau, \xi)\, d\xi + I_i^{(1)}(t, x) \end{aligned} \tag{2.22}$$

when $\alpha_i = 0$, $\beta_i = 1$. Since $\mathbf{w}^*$ is continuous in $\overline{D}_T$, a ladder argument as that in [10] shows that $\mathbf{w}^* \in \mathscr{C}^{1,2}(D_T) \cap \mathscr{C}(\overline{D}_T)$ and is a classical solution of (2.8). It follows from $\mathbf{u}^{(k)} = e^{\gamma t}\mathbf{w}^{(k)}$ that the sequence $\{\mathbf{u}^{(k)}\}$ governed by (2.7) converges to a unique solution $\mathbf{u}^* = e^{-\gamma t}\mathbf{w}^*$ of the truncated problem (1.1). This proves the theorem. ∎

It is obvious from Theorem 2.1 that if the solution $\mathbf{u}^*$ of the truncated problem (1.1) is in $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ then $\hat{f}(\mathbf{u}^*, \mathbf{J} * \mathbf{u}^*) = f(\mathbf{u}^*, \mathbf{J} * \mathbf{u}^*)$, and therefore $\mathbf{u}^*$ is the solution of the original problem (1.1). We show this in the following theorem.

THEOREM 2.2. *Let $\tilde{\mathbf{u}}, \hat{\mathbf{u}}$ be a pair of coupled upper and lower solutions of* (1.1), *and let $f_i(\cdot, \mathbf{u}, \mathbf{w})$ satisfy the Lipschitz condition* (2.3) *for $i = 1, \dots, N$. Then problem* (1.1) *has a unique solution $\mathbf{u}^*(t, x)$ and $\mathbf{u}^* \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$.*

Proof. By the definition of the truncated function $\hat{f}_i$ it suffices to show that for any finite T the solution $\mathbf{u}^*$ of the truncated problem satisfies $\hat{\mathbf{u}} \le \mathbf{u}^* \le \tilde{\mathbf{u}}$ in $\overline{D}_T$. We prove this by showing that the sequence $\{\mathbf{u}^{(k)}\}$ in Theorem 2.1 satisfies the relation $\hat{\mathbf{u}} \le \mathbf{u}^{(k)} \le \tilde{\mathbf{u}}$ in $\overline{D}_T$ for all $k = 1, 2, \dots$ when it holds at $k = 0$.

Let $z_i^{(1)} = \tilde{u}_i - u_i^{(1)}$, $i = 1, \dots, N$. By (2.7), (2.1), and $\hat{\mathbf{u}} \le \mathbf{u}^{(0)} \le \tilde{\mathbf{u}}$, z_i satisfies the boundary-initial relation

$$\begin{aligned} B_i z_i^{(1)} &= B_i \tilde{u}_i - h_i(t, x) \ge 0 && \text{on } S_T \\ z_i^{(1)}(t, x) &= \tilde{u}_i(t, x) - \eta_i(t, x) \ge 0 && \text{in } Q_0^{(i)} \end{aligned} \tag{2.23}$$

and the differential relation

$$\begin{aligned} \mathscr{L}_i z_i^{(1)} &= \left(\partial \tilde{u}_i / \partial t - L_i \tilde{u}_i + K_i \tilde{u}_i \right) - \left(K_i u_i^{(0)} + \hat{f}_i(\mathbf{u}^{(0)}, \mathbf{J} * \mathbf{u}^{(0)}) \right) \\ &\ge \left(f_i(\mathbf{v}, \mathbf{w}) + K_i \tilde{u}_i \right) - \left(K_i u_i^{(0)} + f_i(\mathbf{u}^{(0)}, \mathbf{J} * \mathbf{u}^{(0)}) \right) \end{aligned} \tag{2.24}$$

for any $\mathbf{v}, \mathbf{w} \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ with $v_i = \tilde{u}_i$. Since $\mathbf{u}^{(0)} \le \tilde{\mathbf{u}}$ the relation (2.6) with $\mathbf{v} = (\mathbf{u}^{(0)})^i$ and $\mathbf{w} = \mathbf{J} * \mathbf{u}^{(0)}$ imply that the right-hand side of (2.24) is nonnegative. It follows from the positivity lemma for parabolic boundary-value problems that $z_i^{(1)} \ge 0$ in $\overline{D}_T$ (cf. [9]). This leads to $\mathbf{u}^{(1)} \le \tilde{\mathbf{u}}$. A similar argument using the property of a lower solution gives $\mathbf{u}^{(1)} \ge \hat{\mathbf{u}}$.

Assume, by induction that $\hat{\mathbf{u}} \le \mathbf{u}^{(k)} \le \tilde{\mathbf{u}}$ in $\overline{D}_T$ for some $k > 1$. Then by (2.7) and (2.1), $z_i^{(k+1)} \equiv \tilde{u}_i - u_i^{(k+1)}$ satisfies the relation

$$\mathscr{L}_i z_i^{(k+1)} \ge \left(f_i(\mathbf{v}, \mathbf{w}) + K_i \tilde{u}_i \right) - \left(K_i u_i^{(k)} + f_i(\mathbf{u}^{(k)}, \mathbf{J} * \mathbf{u}^{(k)}) \right)$$

for all $\mathbf{v}, \mathbf{w} \in \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ with $v_i = \tilde{u}_i$. It follows again from (2.6) with $\mathbf{v} = (\mathbf{u}^{(k)})^i$ and $\mathbf{w} = \mathbf{J} * \mathbf{u}^{(k)}$ that the right-hand side of the above inequality is nonnegative. Since $z_i^{(k+1)}$ satisfies the boundary-initial condition (2.23), the positivity lemma ensures that $z_i^{(k+1)} \ge 0$ in $\overline{D}_T$. This yields $\mathbf{u}^{(k+1)} \le \tilde{\mathbf{u}}$. A similar argument gives $\mathbf{u}^{(k+1)} \ge \hat{\mathbf{u}}$. By the induction principle, $\hat{\mathbf{u}} \le \mathbf{u}^{(k)} \le \tilde{\mathbf{u}}$ for every k. The convergence of $\{\mathbf{u}^{(k)}\}$ to $\mathbf{u}^*$ ensures that $\hat{\mathbf{u}} \le \mathbf{u}^* \le \tilde{\mathbf{u}}$. This proves the theorem. ∎

3. DYNAMICS OF THE SYSTEM

In this section we investigate the dynamics of the parabolic system (1.1) in relation to its corresponding steady-state problem

$$\begin{aligned} -L_i u_i &= f_i(x, \mathbf{u}, \mathbf{u}) \quad && \text{in } \Omega \\ B_i u_i &= h_i(x) && \text{on } \partial\Omega \end{aligned} \qquad (i = 1, \ldots, N). \tag{3.1}$$

The following hypothesis will be assumed.

(H) (i) The coefficients of L_i, B_i and the functions f_i, h_i in (1.1) are all independent of t.

(ii) The function $\mathbf{f}(\cdot, \mathbf{u}, \mathbf{v}) \equiv (f_1(\cdot, \mathbf{u}, \mathbf{v}), \ldots, f_N(\cdot, \mathbf{u}, \mathbf{v}))$ is a C^1-function of $\mathbf{u}, \mathbf{v}$ and is mixed quasimonotone in a subset Λ of $\mathbb{R}^N$.

Recall that by writing vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^N$ in the split forms

$$\mathbf{u} \equiv (u_i, [\mathbf{u}]_{a_i}, [\mathbf{u}]_{b_i}), \qquad \mathbf{v} \equiv ([\mathbf{v}]_{c_i}, [\mathbf{v}]_{d_i}),$$

where $[\mathbf{u}]_\sigma$ denotes a vector with σ components of $\mathbf{u}$, the function $\mathbf{f}(\cdot, \mathbf{u}, \mathbf{v})$ is said to have a mixed quasimonotone property in Λ if for each $i = 1, \ldots, N$, there exist nonnegative integers a_i, b_i, c_i and d_i with $a_i + b_i = N - 1$ and $c_i + d_i = N$ such that $f_i(\cdot, \mathbf{u}, \mathbf{v})$ is monotone nondecreasing in $[\mathbf{u}]_{a_i}, [\mathbf{v}]_{c_i}$ and is monotone nonincreasing in $[\mathbf{u}]_{b_i}$, $[\mathbf{v}]_{d_i}$ for all $\mathbf{u}, \mathbf{v} \in \Lambda$ (cf. [9, 10]). If $b_i = d_i = 0$ for all i then $\mathbf{f}(\cdot, \mathbf{u}, \mathbf{v})$ is said to be quasimonotone nondecreasing in Λ. It is easy to see that if $\mathbf{f}(\cdot, \mathbf{u}, \mathbf{v})$ is mixed quasimonotone in the sector $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ then the differential inequalities for $\tilde{\mathbf{u}}, \hat{\mathbf{u}}$ in (2.1) are equivalent to

$$\begin{aligned} \partial \tilde{u}_i / \partial t - L_i \tilde{u}_i &\geq f_i(x, \tilde{u}_i, [\tilde{\mathbf{u}}]_{a_i}, [\hat{\mathbf{u}}]_{b_i}, [\mathbf{J} * \tilde{\mathbf{u}}]_{c_i}, [\mathbf{J} * \hat{\mathbf{u}}]_{d_i}) \\ \partial \hat{u}_i / \partial t - L_i \hat{u}_i &\leq f_i(x, \hat{u}_i, [\hat{\mathbf{u}}]_{a_i}, [\tilde{\mathbf{u}}]_{b_i}, [\mathbf{J} * \hat{\mathbf{u}}]_{c_i}, [\mathbf{J} * \tilde{\mathbf{u}}]_{d_i}). \end{aligned} \tag{3.2}$$

The boundary and initial inequalities for $\tilde{\mathbf{u}}, \hat{\mathbf{u}}$ remain the same as that in (2.1). For the elliptic system (3.1) the requirements of coupled upper and lower solutions, denoted by $\tilde{\mathbf{u}}_s \equiv (\tilde{u}_1, \ldots, \tilde{u}_N)$ and $\hat{\mathbf{u}}_s \equiv (\hat{u}_1, \ldots, \hat{u}_N)$, respectively, are given by $\tilde{\mathbf{u}}_s \geq \hat{\mathbf{u}}_s$ in $\overline{\Omega}$ and

$$\begin{aligned} &\left.\begin{aligned} -L_i \tilde{u}_i &\geq f_i(x, \tilde{u}_i, [\tilde{\mathbf{u}}]_{a_i}, [\hat{\mathbf{u}}]_{b_i}, [\tilde{\mathbf{u}}]_{c_i}, [\hat{\mathbf{u}}]_{d_i}) \\ -L_i \hat{u}_i &\leq f_i(x, \hat{u}_i, [\hat{\mathbf{u}}]_{a_i}, [\tilde{\mathbf{u}}]_{b_i}, [\hat{\mathbf{u}}]_{c_i}, [\tilde{\mathbf{u}}]_{d_i}) \end{aligned}\right\} \quad \text{in } \Omega. \\ &B_i \tilde{u}_i \geq h_i(x) \geq B_i \hat{u}_i \qquad \text{on } \partial\Omega \ (i = 1, \ldots, N). \end{aligned} \tag{3.3}$$

Notice that the requirement of the differential inequalities in (3.3) is more stringent than that for standard elliptic systems when $d_i \neq 0$ (cf. [9]). Since for $u_i \equiv u_i(x)$

$$J_i * u_i = \int_{-\infty}^{t} J_i(t-s,x)u_i(x)\,ds = \left(\int_0^{\infty} J_i(s,x)\,ds\right)u_i(x) = u_i(x),$$

we see from (3.2) and (3.3) that every pair of upper and lower solutions of (3.1) is also a pair of upper and lower solutions of (1.1) whenever $\hat{\mathbf{u}}_s \le \boldsymbol{\eta} \le \tilde{\mathbf{u}}_s$ in $\overline{Q}_0$. By Theorem 2.1, the sector $\langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$ is an invariant set of (1.1) in the sense that if $\boldsymbol{\eta}(t,\cdot) \in \langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$ for $t \le 0$ then the corresponding solution $\mathbf{u}(t,\cdot)$ remains in $\langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$ for all $t > 0$. The aim of this section is to establish a global attractor for the system (1.1) (with respect to $\langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$) and to give a sufficient condition for the asymptotic stability of a steady-state solution.

Given a pair of coupled upper and lower solutions $\tilde{\mathbf{u}}_s, \hat{\mathbf{u}}_s$ of (3.1) we can construct two sequences $\{\overline{\mathbf{u}}_s^{(m)}\} \equiv \{\overline{u}_1^{(m)}, \dots, \overline{u}_N^{(m)}\}$, $\{\underline{\mathbf{u}}_s^{(m)}\} \equiv \{\underline{u}_1^{(m)}, \dots, \underline{u}_N^{(m)}\}$ from the uncoupled linear iteration process

$$\begin{aligned}
-L_i\overline{u}_i^{(m)} + K_i\overline{u}_i^{(m)} &= K_i\overline{u}_i^{(m-1)} \\
&\quad + f_i\Big(x, \overline{u}_i^{(m-1)}, [\overline{\mathbf{u}}^{(m-1)}]_{a_i}, \big[\underline{\mathbf{u}}^{(m-1)}\big]_{b_i}, [\overline{\mathbf{u}}^{(m-1)}]_{c_i}, \big[\underline{\mathbf{u}}^{(m-1)}\big]_{d_i}\Big) \\
-L_i\underline{u}_i^{(m)} + K_i\underline{u}_i^{(m)} &= K_i\underline{u}_i^{(m-1)} \\
&\quad + f_i\Big(x, \underline{u}_i^{(m-1)}, \big[\underline{\mathbf{u}}^{(m-1)}\big]_{a_i}, [\overline{\mathbf{u}}^{(m-1)}]_{b_i}, \big[\underline{\mathbf{u}}^{(m-1)}\big]_{c_i}, [\overline{\mathbf{u}}^{(m-1)}]d_{\mathrm{i}}\Big) \\
B_i\overline{u}_i^{(m)} &= B_i\underline{u}_i^{(m)} = h_i(x) \qquad (i = 1, \dots, N),
\end{aligned} \tag{3.4}$$

where $m = 1, 2, \dots, \overline{\mathbf{u}}^{(0)} = \tilde{\mathbf{u}}_s$, $\underline{\mathbf{u}}^{(0)} = \hat{\mathbf{u}}_s$, and K_i is the Lipschitz constant in (2.3). It is known that if the hypothesis (H) holds with $\Lambda \equiv \langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$ then the sequences $\{\overline{\mathbf{u}}^{(m)}\}$, $\{\underline{\mathbf{u}}^{(m)}\}$ possess the monotone property

$$\hat{\mathbf{u}}_s \le \mathbf{u}_s^{(m)} \le \underline{\mathbf{u}}_s^{(m+1)} \le \overline{\mathbf{u}}_s^{(m+1)} \le \tilde{\mathbf{u}}_s \qquad \text{in } \overline{\Omega},$$

and that the limits

$$\overline{\mathbf{u}}_s(x) \equiv \lim \overline{\mathbf{u}}_s^{(m)}(x), \qquad \underline{\mathbf{u}}_s(x) \equiv \lim \underline{\mathbf{u}}_s^{(m)}(x) \qquad \text{as } m \to \infty$$

exist and satisfy the equations

$$\left.\begin{aligned}
-L_i\overline{u}_i &= f_i\big(x, \overline{u}_i, [\overline{\mathbf{u}}_s]_{a_i}, [\underline{\mathbf{u}}_s]_{b_i}, [\overline{\mathbf{u}}_s]_{c_i}, [\underline{\mathbf{u}}_s]_{d_i}\big) \\
-L_i\underline{u}_i &= f_i\big(x, \underline{u}_i, [\underline{\mathbf{u}}_s]_{a_i}, [\overline{\mathbf{u}}_s]_{b_i}, [\underline{\mathbf{u}}_s]_{c_i}, [\overline{\mathbf{u}}_s]_{d_i}\big)
\end{aligned}\right\} \text{ in } \Omega$$

$$B_i\overline{u}_i = B_i\underline{u}_i = h_i(x) \qquad \text{on } \partial\Omega \ (i = 1, \dots, N), \tag{3.5}$$

where $\bar{\mathbf{u}}_s \equiv (\bar{u}_1, \dots, \bar{u}_N)$ and $\underline{\mathbf{u}}_s \equiv (\underline{u}_1, \dots, \underline{u}_N)$ (cf. [9, 11]). The functions $\bar{\mathbf{u}}_s, \underline{\mathbf{u}}_s$, called quasisolutions of (3.1) in $\langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$, are in general not true solutions. However, if $\bar{\mathbf{u}}_s = \underline{\mathbf{u}}_s$ then $\bar{\mathbf{u}}_s$ is the unique solution of (3.1) in $\langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$. On the other hand, if $\mathbf{f}$ is quasimonotone nondecreasing in $\langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$ (that is, $b_i = d_i = 0$ for all i) then $\bar{\mathbf{u}}_s$ and $\underline{\mathbf{u}}_s$ are true solutions and are called maximal and minimal solutions of (3.1) in $\langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$ (cf. [9]). We show that in either case the sector

$$\langle \underline{\mathbf{u}}_s, \bar{\mathbf{u}}_s \rangle \equiv \left\{ \mathbf{u} \in \mathscr{C}(\overline{\Omega}); \underline{\mathbf{u}}_s \le \mathbf{u} \le \bar{\mathbf{u}}_s \right\}$$

is an attractor of the system (1.1) for $\boldsymbol{\eta} \in \langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$, and under some additional conditions it is a global attraction of (1.1).

To achieve the above goal we imbed the system (1.1) into an extended system of $2N$ equations in the form

$$\begin{aligned} &\left.\begin{aligned} \partial u_i/\partial t - L_i u_i &= F_i(x, \mathbf{u}, \mathbf{v}, \mathbf{J}*\mathbf{u}, \mathbf{J}*\mathbf{v}) \\ \partial v_i/\partial t - L_i v_i &= G_i(x, \mathbf{u}, \mathbf{v}, \mathbf{J}*\mathbf{u}, \mathbf{J}*\mathbf{v}) \end{aligned}\right. \quad (t > 0,\ x \in \Omega) \\ &B_i u_i = h_i(x), \qquad B_i v_i = p_i(x) \ (t > 0,\ x \in \partial\Omega) \\ &u_i(t, x) = \eta_i(t, x), \qquad v_i(t, x) = \nu_i(t, x) \text{ in } Q_0 \ (i = 1, \dots, N), \end{aligned} \tag{3.6}$$

where

$$\begin{aligned} &F_i(x, \mathbf{u}, \mathbf{v}, \mathbf{J}*\mathbf{u}, \mathbf{J}*\mathbf{v}) \equiv f_i\big(x, u_i, [\mathbf{u}]_{a_i}, [\mathbf{M} - \mathbf{v}]_{b_i}, [J*\mathbf{u}]_{c_i}, [\mathbf{J}*(\mathbf{M} - \mathbf{v})]_{d_i}\big) \\ &G_i(x, \mathbf{u}, \mathbf{v}, \mathbf{J}*\mathbf{u}, \mathbf{J}*\mathbf{v}) \\ &\qquad \equiv -f_i\big(x, M_i - v_i, [\mathbf{M} - \mathbf{v}]_{a_i}, [\mathbf{u}]_{b_i}, [\mathbf{J}*(\mathbf{M} - \mathbf{v})]_{c_i}, [\mathbf{J}*\mathbf{u}]_{d_i}\big) \\ &p_i(x) \equiv M_i \beta_i - h_i(x), \qquad \nu_i(t, x) \equiv M_i - \eta_i(t, x) \ (i = 1, \dots, N), \end{aligned} \tag{3.7}$$

and $\mathbf{M} \equiv (M_1, \dots, M_N)$ is any constant vector satisfying $\mathbf{M} \ge \tilde{\mathbf{u}}_s$ in $\overline{\Omega}$. Similarly, the corresponding extended elliptic system of (3.1) is given by

$$\begin{aligned} &\left.\begin{aligned} -L_i u_i &= F_i(x, \mathbf{u}, \mathbf{v}, \mathbf{u}, \mathbf{v}) \\ -L_i v_i &= G_i(x, \mathbf{u}, \mathbf{v}, \mathbf{u}, \mathbf{v}) \end{aligned}\right. \quad (t \in \Omega) \\ &B_i u_i = h_i(x), \qquad B_i v_i = p_i(x) \ (x \in \partial\Omega) \ (i = 1, \dots, N). \end{aligned} \tag{3.8}$$

It is easy to see from the mixed quasimonotone property of $\mathbf{f} \equiv (f_1, \dots, f_N)$ that the $2N$-vector function

$$\begin{aligned} &\mathscr{F}(\cdot, \mathbf{w}, \mathbf{J}*\mathbf{w}) \\ &\quad \equiv \big(F_1(\mathbf{w}, \mathbf{J}*\mathbf{w}), \dots, F_N(\mathbf{w}, \mathbf{J}*\mathbf{w}), G_1(\mathbf{w}, \mathbf{J}*\mathbf{w}), \dots, G_N(\mathbf{w}, \mathbf{J}*\mathbf{w})\big) \end{aligned} \tag{3.9}$$

is quasimonotone nondecreasing in $\langle \hat{\mathbf{w}}_s, \tilde{\mathbf{w}}_s \rangle$, where

$$\begin{aligned}
\tilde{\mathbf{w}}_s &\equiv (\tilde{\mathbf{u}}_s, \tilde{\mathbf{v}}_s) \equiv \left(\tilde{u}_1, \dots, \tilde{u}_N, M_1 - \hat{u}_1, \dots, M_N - \hat{u}_N\right) \\
\hat{\mathbf{w}}_s &\equiv (\hat{\mathbf{u}}_s, \hat{\mathbf{v}}_s) \equiv \left(\hat{u}_1, \dots, \hat{u}_N, M_1 - \tilde{u}_1, \dots, M_N - \tilde{u}_N\right) \qquad (3.10) \\
\langle \hat{\mathbf{w}}_s, \tilde{\mathbf{w}}_s \rangle &\equiv \left\{\mathbf{w} \equiv (\mathbf{u}, \mathbf{v}); \hat{\mathbf{w}}_s \le \mathbf{w} \le \tilde{\mathbf{w}}_s\right\}.
\end{aligned}$$

Moreover, we have the following results from [11].

THEOREM A. *Let hypothesis* (H) *hold with* $\Lambda \equiv \langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$. *Then* (i) $\tilde{\mathbf{u}}_s$ *and* $\hat{\mathbf{u}}_s$ *are coupled upper and lower solutions of* (3.1) *if and only if* $(\tilde{\mathbf{u}}_s, \mathbf{M} - \hat{u}_s)$ *and* $(\hat{\mathbf{u}}_s, \mathbf{M} - \tilde{\mathbf{u}}_s)$ *are ordered upper and lower solutions of* (3.8), (ii) $\bar{\mathbf{u}}_s$ *and* $\underline{\mathbf{u}}_s$ *are the quasisolutions of* (3.1) *in* $\langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$ *if and only if* $(\bar{\mathbf{u}}_s, \mathbf{M} - \underline{\mathbf{u}}_s)$ *and* $(\underline{\mathbf{u}}_s, \mathbf{M} - \bar{\mathbf{u}}_s)$ *are the maximal and minimal solutions of* (3.8) *in* $\langle \hat{\mathbf{w}}_s, \tilde{\mathbf{w}}_s \rangle$, *and* (iii) $\bar{\mathbf{u}}_s$ *is the unique solution of* (3.1) *and* $(\bar{\mathbf{u}}_s, \mathbf{M} - \underline{\mathbf{u}}_s)$ *is the unique solution of* (3.8) *if* $\bar{\mathbf{u}}_s = \underline{\mathbf{u}}_s$.

For the parabolic systems (3.1) *and* (3.6) *we have*

LEMMA 3.1. *Let* $\tilde{\mathbf{u}}, \hat{\mathbf{u}}$ *be a pair of coupled upper and lower solutions of* (1.1), *and let hypothesis* (H) *hold with* $\Lambda \equiv \langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$. *Then* $\mathbf{u}$ *is the unique solution of* (1.1) *in* $\langle \hat{\mathbf{u}}, \tilde{\mathbf{u}} \rangle$ *if and only if* $(\mathbf{u}, \mathbf{M} - \mathbf{u})$ *is the unique solution of* (3.6) *in* $\langle \hat{\mathbf{w}}, \tilde{\mathbf{w}} \rangle$, *where* $\tilde{\mathbf{w}} = \mathbf{M} - \hat{\mathbf{u}}$, $\hat{\mathbf{w}} = \mathbf{M} - \tilde{\mathbf{u}}$.

Proof. The proof follows from the existence and uniqueness result in Theorem 2.2 and the equivalence relation between the systems (1.1) and (3.6). We omit the details. ∎

In view of Lemma 3.1 the dynamics of the system (1.1) can be determined by the solution property of (3.6). Since the reaction function $\mathscr{F}$ in (3.9) is quasimonotone nondecreasing the asymptotic behavior of the solution $(\mathbf{u}, \mathbf{v})$ can be investigated by using some special initial function $(\boldsymbol{\eta}, \boldsymbol{\nu})$. For this purpose we first establish a positivity lemma for a function $\mathbf{z} \equiv (z_1, \dots, z_M)$ satisfying the relation

$$\begin{aligned}
\partial z_i/\partial t - L_i z_i &\ge \sum_{j=1}^{M} b_{ij} z_j + \sum_{j=1}^{M} c_{ij}\left(J_j * z_j\right) \qquad \text{in } D_T \\
B_i z_i &\ge 0 \qquad \text{on } S_T \\
z_i(t, x) &\ge 0 \qquad \text{in } Q_0^{(i)}\ (i = 1, \dots, M), \qquad (3.11)
\end{aligned}$$

where $b_{ij} \equiv b_{ij}(t,x)$, $c_{ij} \equiv c_{ij}(t,x)$ are in $C(\overline{D}_T)$, and

$$J_j * z_j \equiv \begin{cases} \int_{-\infty}^{t} J_j(t-s,x) z_j(s,x)\, ds & \text{for } j = 1,\dots,m_0, \\ z_j(t-r_j,x) & \text{for } j = m_0+1,\dots,M. \end{cases} \tag{3.12}$$

LEMMA 3.2. *Let $b_{ij}, c_{ij} \in C(\overline{D}_T)$ such that $b_{ij} \geq 0$ for $j \neq i$ and $c_{ij} \geq 0$ for all $i,j = 1,\dots,M$, and let $\mathbf{z} \equiv (z_1,\dots,z_M) \in \mathscr{C}^{1,2}(D_T) \cap \mathscr{C}(\overline{D}_T)$ and satisfy the inequalities in* (3.11). *Then $\mathbf{z} \geq 0$ in $\overline{D}_T$.*

Proof. A proof for the case with only discrete delay has been given in [12]. For the case with continuous delay we need to use a different approach. By the relation (3.11), $\mathbf{z} \equiv (z_1,\dots,z_M)$ satisfies the linear system

$$\begin{aligned} \partial z_i/\partial t - L_i z_i &= \sum_{j=1}^{M} b_{ij} z_j + \sum_{j=1}^{M} c_{ij}(J_j * z_j) + q_i(t,x) \qquad \text{in } D_T \\ B_i z_i &= g_i(t,x) \qquad \text{on } S_T \\ z_i(t,x) &= \theta_i(t,x) \qquad \text{in } Q_0^{(i)}\ (i = 1,\dots,M), \end{aligned} \tag{3.13}$$

where q_i, g_i, and θ_i are some continuous nonnegative functions in their respective domains. It is obvious that the above system may be considered as a special case of (1.1) with $u_i = z_i$, $N = M$, and f_i given by the right-hand side of the differential equation in (3.13). Consider the case where the functions b_{ij}, c_{ij}, and the data q_i, g_i, and θ_i are Holder continuous in their respective domains. Since by the hypothesis on b_{ij} and c_{ij}, the function $\mathbf{f} \equiv (f_1,\dots,f_M)$ in (3.13) is quasimonotone nondecreasing in $\mathbb{R}^M$, Theorem 2.2 ensures that problem (3.13) has a unique nonnegative solution if there exist a pair of ordered nonnegative upper and lower solutions. It is clear from the nonnegative property of q_i, g_i, and θ_i that $\hat{\mathbf{z}} = \mathbf{0}$ is a lower solution. To find a positive upper solution $\tilde{\mathbf{z}} \equiv (\tilde{z}_1,\dots,\tilde{z}_M)$ we seek it in the form

$$\tilde{z}_i = y_i + \rho e^{\alpha t} \qquad (i = 1,\dots,M),$$

where $\rho \geq 1$ and $\alpha \geq 1$ are some constants to be chosen and $y_i \equiv y_i(t,x)$ is the nonnegative solution of the linear equation

$$\partial y_i/\partial t - L_i y_i = q_i(t,x) \qquad \text{in } D_T$$

under the same boundary and initial conditions as that in (3.13). It is easily seen that $\tilde{z}$ satisfies the boundary and initial inequalities of an upper solution. This implies that $\tilde{\mathbf{z}}$ is an upper solution if

$$\alpha\rho e^{\alpha t} \geq \sum_{j=1}^{M} b_{ij}\left(y_j + \rho e^{\alpha t}\right) + \sum_{j=1}^{M} c_{ij} J_j * \left(y_j + \rho e^{\alpha t}\right) \tag{3.14}$$

for every i. Since by (1.4)–(1.7),

$$J_j * (\rho e^{\alpha t}) \equiv \rho \int_{-\infty}^{t} J_j(t-s,x) e^{\alpha s}\, ds = \rho \int_0^{\infty} J_j(s,x) e^{\alpha(t-s)}\, ds \leq \rho e^{\alpha t}$$

(and similarly for finite continuous delay) when $j = 1, \ldots, m_0$, and

$$J_j * (\rho e^{\alpha t}) = \rho e^{\alpha(t-r_j)} \leq \rho e^{\alpha t}$$

when $j = m_0 + 1, \ldots, M$, we see from $\rho \geq 1$, $\alpha \geq 1$ that relation (3.14) holds if

$$\alpha \geq \sum_{j=1}^{M} b_{ij}\left(y_j + 1\right) + \sum_{j=1}^{M} c_{ij} J_j * \left(y_j + 1\right).$$

This inequality is clearly satisfied by a sufficiently large α. With this choice of α and any $\rho \geq 1$, $\tilde{\mathbf{z}}$ is a positive upper solution of (3.13). By Theorem 2.2, problem (3.13) has a unique nonnegative solution $\mathbf{z}^*$ in $\langle \mathbf{0}, \tilde{\mathbf{z}} \rangle$. Since $\mathbf{z} \leq \rho e^{\alpha t}$ for some ρ and α, the uniqueness property of the solution ensures that $\mathbf{z} = \mathbf{z}^* \geq \mathbf{0}$ in $\overline{D}_T$.

If b_{ij}, c_{ij}, and the data are only continuous functions we may use the integral representations (2.21)–(2.22) for the solution $\mathbf{z}$ of (3.13). Since the function $\mathbf{f} \equiv (f_1, \ldots, f_M)$ in (3.13) is globally Lipschitz continuous the proof of Theorem 2.1 shows that $\mathbf{z}$ is the unique solution of the integral equation, and for any initial iteration $\mathbf{z}^{(0)} \in \mathscr{C}(\overline{D}_T)$ the sequence $\{\mathbf{z}^{(m)}\}$ given by (2.20) (with $\mathbf{w}^{(m)}$ replaced by $\mathbf{z}^{(m)}$) converges in $\mathscr{C}(\overline{D}_T)$ to $\mathbf{z}$. It follows by using any $\mathbf{z}^{(0)} \in \langle \mathbf{0}, \tilde{\mathbf{z}} \rangle$ as the initial iteration the argument in the proof of Theorem 2.2 for the corresponding integral equations shows that $\mathbf{z} \in \langle \mathbf{0}, \tilde{\mathbf{z}} \rangle$. This leads to $\mathbf{z} \geq \mathbf{0}$ in $\overline{D}_T$. ∎

Given a pair of coupled upper and lower solutions $\tilde{\mathbf{u}}_s$, $\hat{\mathbf{u}}_s$ of (3.1) Theorem A implies that $(\tilde{\mathbf{u}}_s, \mathbf{M} - \hat{\mathbf{u}}_s)$ and $(\hat{\mathbf{u}}_s, \mathbf{M} - \tilde{\mathbf{u}}_s)$ are ordered upper and lower solutions of (3.8). Since the latter pair are also upper and lower solutions of (3.6) when $\hat{\mathbf{u}}_s \leq \boldsymbol{\eta} \leq \tilde{\mathbf{u}}_s$ and $\mathbf{M} - \tilde{\mathbf{u}}_s \leq \boldsymbol{\nu} \leq \mathbf{M} - \hat{\mathbf{u}}_s$, Theorem 2.2 ensures that the extended problem (3.6) has a unique solution $(\mathbf{u}, \mathbf{v})$ when $\hat{\mathbf{u}}_s \leq \boldsymbol{\eta} \leq \tilde{\mathbf{u}}_s$ and $\hat{\mathbf{u}}_s \leq \mathbf{M} - \boldsymbol{\nu} \leq \tilde{\mathbf{u}}_s$. The following theorem gives the monotone convergence of the solution $(\mathbf{u}, \mathbf{v})$ when $(\boldsymbol{\eta}, \boldsymbol{\nu})$ is taken either as $(\tilde{\mathbf{u}}_s, \mathbf{M} - \hat{\mathbf{u}}_s)$ or as $(\hat{\mathbf{u}}_s, \mathbf{M} - \tilde{\mathbf{u}}_s)$.

THEOREM 3.1. *Let* $\tilde{\mathbf{u}}_s$, $\hat{\mathbf{u}}_s$ *be a pair of coupled upper and lower solutions of* (3.1), *and let hypothesis* (H) *hold with* $\Lambda \equiv \langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$. *Denote by* $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ *and* $(\underline{\mathbf{u}}, \underline{\mathbf{v}})$ *the solutions of* (3.6) *with* $(\boldsymbol{\eta}, \boldsymbol{\nu}) = (\tilde{\mathbf{u}}_s, \mathbf{M} - \hat{\mathbf{u}}_s)$ *and* $(\boldsymbol{\eta}, \boldsymbol{\nu}) = (\hat{\mathbf{u}}_s, \mathbf{M} - \tilde{\mathbf{u}}_s)$, *respectively. Then* $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ *converges monotonically from above to the maximal solution* $(\bar{\mathbf{u}}_s, \bar{\mathbf{v}}_s)$ *of* (3.8) *as* $t \to \infty$, *and* $(\underline{\mathbf{u}}, \underline{\mathbf{v}})$ *converges monotonically from below to the minimal solution* $(\underline{\mathbf{u}}_s, \underline{\mathbf{v}}_s)$. *Moreover*, $\bar{\mathbf{v}}_s = \mathbf{M} - \underline{\mathbf{u}}_s, \underline{\mathbf{v}}_s = \mathbf{M} - \bar{\mathbf{u}}_s$, *and the pair* $\bar{\mathbf{u}}_s$, $\underline{\mathbf{u}}_s$ *satisfy the equations in* (3.5).

Proof. We prove the theorem by following the same idea as that in [12] using the result of Lemma 3.2. Let $\delta > 0$ be an arbitrary constant, and let

$$\underline{z}_i(t,x) = \underline{w}_i(t+\delta, x) - \underline{w}_i(t,x) \qquad \text{for } i = 1, \dots, 2N,$$

where

$$\underline{w} \equiv (\underline{w}_1, \dots, \underline{w}_N, \underline{w}_{N+1}, \dots, \underline{w}_{2N}) \equiv (\underline{u}_1, \dots, \underline{u}_N, \underline{v}_1, \dots, \underline{v}_N),$$

and $\underline{u}_i$ and $\underline{v}_i$ are the respective components of $\underline{\mathbf{u}}$ and $\underline{\mathbf{v}}$. By (3.6), (3.9), and $(\underline{\mathbf{u}}(t,x), \underline{\mathbf{v}}(t,x)) = (\hat{\mathbf{u}}_s(x), \mathbf{M} - \tilde{\mathbf{u}}_s(x))$ in $Q^{(0)}$,

$$\begin{aligned} \partial \underline{z}_i/\partial t - L_i \underline{z}_i &= F_i(\mathbf{w}(t+\delta, x), (\mathbf{J} * \mathbf{w})(t+\delta, x)) \\ &\quad - F_i(\mathbf{w}(t,x), (\mathbf{J} * \mathbf{w})(t,x)) \qquad \text{in } D_T \\ B_i \underline{z}_i &= 0 \qquad \text{on } S_T \\ \underline{z}_i(t,x) &\ge 0 \qquad \text{in } Q_0^{(i)} \ (i = 1, \dots, 2N), \end{aligned} \tag{3.15}$$

where $F_{N+i}(\mathbf{w}, \mathbf{J} * \mathbf{w}) = G_i(\mathbf{w}, \mathbf{J} * \mathbf{w})$ for $i = 1, \dots, N$. By the mean-value theorem, there exist intermediate values $\bar{\xi}_l \equiv \bar{\xi}_l(t,x)$, $l = 1, 2$, such that the differential relation in (3.15) becomes

$$\partial \underline{z}_i/\partial t - L_i \underline{z}_i = \sum_{j=1}^{2N} b_{ij} \underline{z}_j + \sum_{j=1}^{2N} c_{ij} J_j * \underline{z}_j,$$

where for each $i = 1, \dots, 2N$,

$$b_{ij} \equiv (\partial F_i/\partial w_j)(\boldsymbol{\xi}_1), \qquad c_{ij} \equiv \big(\partial F_i/\partial(J_j * w_j)\big)(\boldsymbol{\xi}_2), \ j = 1, \dots, 2N.$$

Since by the quasimonotone nondecreasing property of the function $\mathscr{F}(\mathbf{w}, \mathbf{J} * \mathbf{w})$, $b_{ij} \ge 0$ for $j \ne i$ and $c_{ij} \ge 0$ for all $i, j = 1, \dots, 2N$ we see from Lemma 3.2 (with $M = 2N$) that $\underline{z}_i \ge 0$ for $i = 1, \dots, 2N$. This leads to $\underline{\mathbf{u}}(t+\delta, x) \ge \underline{\mathbf{u}}(t,x)$ and $\underline{\mathbf{v}}(t+\delta, x) \ge \underline{\mathbf{v}}(t,x)$ in $\bar{D}_T$. The arbitrariness of $\delta > 0$ and $T < \infty$ implies that for every $x \in \bar{\Omega}$, $(\underline{\mathbf{u}}(t,x), \underline{\mathbf{v}}(t,x))$ is monotone nondecreasing in $t > 0$. A similar argument shows that $(\bar{\mathbf{u}}(t,x), \underline{\mathbf{v}}(t,x))$ is monotone nonincreasing in $t > 0$ and $(\bar{\mathbf{u}}(t,x), \bar{\mathbf{v}}(t,x)) \ge (\underline{\mathbf{u}}(t,x),$

$\underline{\mathbf{v}}(t,x))$. It follows from these properties that the pointwise limits

$$\lim_{t\to\infty}\left(\bar{\mathbf{u}}(t,x),\bar{\mathbf{v}}(t,x)\right) \equiv \left(\bar{\mathbf{u}}_s(x),\bar{\mathbf{v}}_s(x)\right),$$

$$\lim_{t\to\infty}\left(\underline{\mathbf{u}}(t,x)\,\underline{\mathbf{v}}(t,x)\right) \equiv \left(\underline{\mathbf{u}}_s(x),\underline{\mathbf{v}}_s(x)\right)$$

exist and satisfy the relation

$$\left(\bar{\mathbf{u}}(t,x),\underline{\mathbf{v}}(t,x)\right) \le \left(\underline{\mathbf{u}}_s(x),\underline{\mathbf{v}}_s(x)\right) \le \left(\bar{\mathbf{u}}_s(x),\bar{\mathbf{v}}_s(x)\right) \le \left(\bar{\mathbf{u}}(t,x),\bar{\mathbf{v}}(t,x)\right)$$

for all $t > 0$, $x \in \overline{\Omega}$. Using the positivity Lemma 3.2, the argument in [9, 12] shows that $(\bar{\mathbf{u}}_s, \bar{\mathbf{v}}_s)$ and $(\underline{\mathbf{u}}_s, \underline{\mathbf{v}}_s)$ are the respective maximal and minimal solutions of (3.8) and $\bar{\mathbf{v}}_s = \mathbf{M} - \underline{\mathbf{u}}_s$, $\underline{\mathbf{v}}_s = \mathbf{M} - \bar{\mathbf{u}}_s$. Finally by Theorem A, $\bar{\mathbf{u}}_s$ and $\underline{\mathbf{u}}_s$ satisfy the equations in (3.5). This proves the theorem. ∎

Based on Theorem 3.1 we have the following dynamic property of the system (1.1) for an arbitrary initial function $\boldsymbol{\eta}$ in $\langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$.

THEOREM 3.2. *Let $\tilde{\mathbf{u}}_s$, $\hat{\mathbf{u}}_s$ be a pair of coupled upper and lower solutions of* (3.1), *and let hypothesis* (H) *hold with $\Lambda \equiv \langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$. Denote by $\bar{\mathbf{u}}_s \equiv (\bar{\mathbf{u}}_1, \dots, \bar{\mathbf{u}}_N)$, $\underline{\mathbf{u}}_s \equiv (\underline{\mathbf{u}}_1, \dots, \underline{\mathbf{u}}_N)$ the quasisolutions of* (3.1) *which satisfy* (3.5). *Then for any initial function $\boldsymbol{\eta}$ satisfying $\hat{\mathbf{u}}_s \le \boldsymbol{\eta} \le \tilde{\mathbf{u}}_s$ in Q_0 the corresponding solution $\mathbf{u}^*(t,x)$ of* (1.1) *possesses the property*

$$\underline{\mathbf{u}}(x) \le \mathbf{u}^*(t,x) \le \bar{\mathbf{u}}(x) \qquad \text{as } t \to \infty \; (x \in \overline{\Omega}). \tag{3.16}$$

Moreover, if $\bar{\mathbf{u}}_s = \underline{\mathbf{u}}_s \equiv \mathbf{u}_s^*$ then $\mathbf{u}_s^*$ is the unique solution of (3.1) *in $\langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$ and $\mathbf{u}^*(t,x) \to \mathbf{u}_s^*(x)$ as $t \to \infty$.*

Proof. Consider the extended problem (3.6) and its corresponding elliptic system (3.8). By Theorem A, the pair $(\bar{\mathbf{u}}_s, \mathbf{M} - \hat{\mathbf{u}}_s)$ and $(\hat{\mathbf{u}}_s, \mathbf{M} - \tilde{\mathbf{u}}_s)$ are ordered upper and lower solutions of (3.8). Since the function $\mathscr{F}(\cdot, \mathbf{w}, \mathbf{J} * \mathbf{w})$ in (3.9) is quasimonotone nondecreasing and the relation $\hat{\mathbf{u}}_s \le \boldsymbol{\eta} \le \tilde{\mathbf{u}}_s$ is equivalent to $\mathbf{M} - \tilde{\mathbf{u}}_s \le \boldsymbol{\nu} \le \mathbf{M} - \hat{\mathbf{u}}_s$ we see that the solutions $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$, $(\underline{\mathbf{u}}, \underline{\mathbf{v}})$ in Theorem 3.1 are ordered upper and lower solutions of (3.6) when $\boldsymbol{\eta} \in \langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$. By an application of Theorem 2.2, problem (3.6) has a unique solution $(\mathbf{u}^*, \mathbf{v}^*)$ such that $(\underline{\mathbf{u}}, \underline{\mathbf{v}}) \le (\mathbf{u}^*, \mathbf{v}^*) \le (\bar{\mathbf{u}}, \bar{\mathbf{v}})$ in $\overline{D}_T$. The arbitrariness of $T < \infty$ ensures that

$$\underline{\mathbf{u}}(t,x) \le \mathbf{u}^*(t,x) \le \bar{\mathbf{u}}(t,x) \qquad \text{in } \mathbb{R}^+ \times \overline{\Omega}. \tag{3.17}$$

Since by Theorem 3.1, $(\bar{\mathbf{u}}, \bar{\mathbf{v}})$ converges to the maximal solution $(\bar{\mathbf{u}}_s, \mathbf{M} - \underline{\mathbf{u}}_s)$ and $(\underline{\mathbf{u}}, \underline{\mathbf{v}})$ converges to the minimal solution $(\underline{\mathbf{u}}_s, \mathbf{M} - \bar{\mathbf{u}}_s)$ as $t \to \infty$, and by Theorem A the pair $\bar{\mathbf{u}}_s$ and $\underline{\mathbf{u}}_s$ are quasisolutions of (3.1) we conclude from (3.17) that $\mathbf{u}^*(t,x)$ satisfies the relation (3.16). Finally, if $\bar{\mathbf{u}}_s = \underline{\mathbf{u}}_s \equiv \mathbf{u}_s^*$ then by Theorem A, $\mathbf{u}_s^*$ is the unique solution of (3.1) in $\langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$, and both $\bar{\mathbf{u}}(t,x)$ and $\underline{u}(t,x)$ converge to $\mathbf{u}_s^*(x)$ as $t \to \infty$. The convergence of $\mathbf{u}^*(t,x)$ to $\mathbf{u}_s^*(x)$ as $t \to \infty$ follows from (3.17). This proves the theorem. ∎

Theorem 3.2 implies that the sector $\langle \underline{\mathbf{u}}_s, \bar{\mathbf{u}}_s \rangle$ between the quasisolutions $\bar{\mathbf{u}}_s$ and $\underline{\mathbf{u}}_s$ is an attractor of (1.1) for all initial functions $\boldsymbol{\eta}$ in $\langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$, and if $\underline{\mathbf{u}}_s = \bar{\mathbf{u}}_s$ then $\underline{\mathbf{u}}_s$ is a steady-state solution which is asymptotically stable with a stability region $\langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$. In the following theorem we show, under some additional conditions, that the sector $\langle \underline{\mathbf{u}}_s, \bar{\mathbf{u}}_s \rangle$ is a global attractor of (1.1) for the case of finite continuous and finite discrete delays.

THEOREM 3.3. *Let the hypotheses in Theorem* 3.2 *hold, and let* $\mathbf{u}^*(t, x)$ *be the solution of* (1.1) *with an arbitrary initial function* $\boldsymbol{\eta}(t, x)$. *Assume that* $J_i * u_i$ *is given by* (1.5) *for* $i = 1, \dots, n_0$. *If there exists a finite* $t^* > 0$ *such that*

$$\hat{u}_i(x) \le u_i^*(t, x) \le \tilde{u}_i(x) \qquad \text{for } t^* - r_i \le t \le t^*,\ x \in \Omega\ (i = 1, \dots, N) \tag{3.18}$$

then $\mathbf{u}^*(t, x)$ *satisfies relation* (3.16). *Moreover*, $\mathbf{u}^*(t, x) \to \mathbf{u}_s^*(x)$ *as* $t \to \infty$ *if* $\bar{\mathbf{u}} = \underline{\mathbf{u}}_s \equiv \mathbf{u}_s^*$.

Proof. Let $\tau = t - t^*$, $U_i(\tau, x) = u_i^*(\tau + t^*, x)$, and consider the system (1.1) with $t = \tau + t^*$. By (1.5), (1.7), and $t^* > 0$,

$$\begin{aligned}(J_i * u_i^*)(t, x) &= \int_0^{r_i + t} J_i(s, x) u_i^*(t - s, x)\, ds \\ &= \int_0^{r_i + \tau} J_i(s, x) U_i(\tau - s, x)\, ds \\ &= \int_{-r_i}^{\tau} J_i(\tau - s, x) U_i(s, x)\, ds = (J_i * U_i)(\tau, x)\end{aligned}$$

when $i = 1, \dots, n_0$, and

$$(J_i * u_i^*)(t, x) = u^*(t - r_i, x) = U(\tau - r_i, x) = (J_i * U_i)(\tau, x)$$

when $i = n_0 + 1, \dots, N$. Using the above relation in (1.1) yields

$$\begin{gathered}\partial U_i / \partial t - L_i U_i = f_i(x, \mathbf{U}, \mathbf{J} * \mathbf{U}) \qquad (\tau > 0,\ x \in \Omega) \\ B_i U_i = h_i(x) \qquad (\tau > 0,\ x \in \partial\Omega) \\ U_i(\tau, x) = u^*(\tau + t^*, x) \qquad (\tau \in I_i,\ x \in \Omega)\ (i = 1, \dots, N),\end{gathered} \tag{3.19}$$

where $\mathbf{U} \equiv (U_1, \dots, U_N)$, $\mathbf{J} * \mathbf{U} \equiv (J_1 * U_1, \dots, J_N * U_N)$, and $I_i = [-r_i, 0]$ for all $i = 1, \dots, N$. It is obvious that the corresponding steady-state problem of (3.19) is also given by (3.1). Since by (3.18)

$$\hat{u}_i(x) \le U_i(\tau, x) \le \tilde{u}_i(x) \qquad \text{for } \tau \in [-r_i, 0] \times \Omega$$

the initial function in (3.19) is in $\langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$ for $\tau \in I_i$. It follows from Theorem 3.2 that $\hat{\mathbf{u}}_s(x) \le \mathbf{U}(\tau, x) \le \tilde{\mathbf{u}}_s(x)$ as $\tau \to \infty$, and if $\bar{\mathbf{u}}_s = \underline{\mathbf{u}}_s \equiv \mathbf{u}_s^*$ then $\mathbf{U}(\tau, x) \to \mathbf{u}_s^*(x)$ as $\tau \to \infty$. This proves the theorem. ∎

Theorem 3.3 implies that for parabolic systems with a combination of finite continuous and finite discrete delays if the solution $\mathbf{u}^*(t, x)$ corresponding to an arbitrary initial function $\boldsymbol{\eta}(t, x)$ (subject to the general conditions given in Section 2) enters the sector $\langle \hat{\mathbf{u}}_s, \tilde{\mathbf{u}}_s \rangle$ for a suitable period of time then it is eventually attracted to the sector $\langle \underline{\mathbf{u}}_s, \overline{\mathbf{u}}_s \rangle$. In particular, if $\hat{\mathbf{u}}_s > 0$ (or $\hat{\mathbf{u}}_s \geq 0$ and $\hat{u}_i \not\equiv 0$ for all $i = 1, \dots, N$) then $\overline{\mathbf{u}}_s \geq \mathbf{u}_s > \mathbf{0}$ and $\langle \underline{\mathbf{u}}_s, \overline{\mathbf{u}}_s \rangle$ is a positive global attractor. This result is very useful for studying the persistence and permanence of various competition and prey-predator systems in ecology.

4. APPLICATION TO SOME MODEL PROBLEMS

To give some applications of the results obtained in Sections 2 and 3 we consider three model problems arising from population dynamics and chemical reactions. In these model problems it is assumed that the function $J_i(t, x)$ possesses the property (1.6) (or (1.7)), the initial function $\boldsymbol{\eta}(t, x)$ is nonnegative in Q_0, and the coefficients of L_i and B_i are independent of t.

(A) *A Diffusive Logistic Model*

The first model problem is concerned with a scalar diffusive logistic model with continuous time delay which is given by

$$\begin{aligned} \partial u/\partial t - Lu &= u(a - bu - cJ * u) && (t > 0,\ x \in \Omega) \\ Bu \equiv \alpha \partial u/\partial \nu + \beta u &= 0 && (t > 0,\ x \in \partial\Omega) \\ u(t, x) &= \eta(t, x) && (t \in I,\ x \in \Omega), \end{aligned} \tag{4.1}$$

where L and B are in the form of (1.2), $a \equiv a(x)$, $b \equiv b(x)$ and $c \equiv c(x)$ are in $C^{\alpha}(\overline{\Omega})$, and $J * u$ is given in the form (1.4) with either $I = (-\infty, 0]$ for infinite delay or $I = [-r, 0]$ for finite delay. Problem (4.1) has been treated in [14] for the care $r = 0$ and Neumann boundary condition; in [15, 18] for $r = 0$ and Dirichlet boundary condition; and in [12] for discrete time delay ($J * u \equiv u(t - r, x)$) and more general boundary condition. The main concern in these works is the asymptotic behavior of the solution in relation to the positive solution of the steady-state problem

$$\begin{aligned} -Lu &= u(a - bu - cu) && \text{in } \Omega \\ Bu &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{4.2}$$

Since problem (4.1) is a special case of (1.1) with $N = 1$ and

$$f(x, u, J * u) = u(a - bu - cJ * u) \tag{4.3}$$

the conditions in hypothesis (H) are all satisfied. Hence to study the asymptotic behavior of the solution of (4.1) by Theorems 3.2 and 3.3 it suffices to construct a suitable pair of upper and lower solutions of (4.2). In view of the mixed quasimonotone property of $f(\cdot, u, J * u)$, upper and lower solutions $\tilde{u}_s$, $\hat{u}_s$ are required to satisfy the relation

$$\left.\begin{aligned} -L\tilde{u}_s &\geq \tilde{u}_s(a - b\tilde{u}_s - c\hat{u}_s) \\ -L\hat{u}_s &\leq \hat{u}_s(a - b\hat{u}_s - c\tilde{u}_s) \end{aligned}\right. \quad \text{in } \Omega$$

$$B\tilde{u}_s \geq 0 \geq B\hat{u}_s \qquad \text{on } \partial\Omega. \tag{4.4}$$

Notice that the above requirement is not the same as that in standard elliptic boundary-value problems.

Let λ_0, $\phi(x)$ be the smallest eigenvalue and its corresponding positive (normalized) eigenfunction of the eigenvalue problem

$$L\phi + \lambda a(x)\phi = 0 \text{ in } \Omega, \qquad B\phi = 0 \text{ on } \partial\Omega. \tag{4.5}$$

It is well known that $\lambda_0 > 0$ when $\beta(x) \not\equiv 0$ and $\lambda_0 = 0$ when $\beta(x) \equiv 0$. Moreover, problem (4.2) has only the trivial solution $u_s = 0$ if $\lambda_0 \geq 1$ and it has a unique positive solution $u_s^*(x)$ if $\lambda_0 < 1$ (cf. [9]). Since for any constant $M \geq a/b$ in Ω the pair $\tilde{u}_s = M$ and $\hat{u}_s = 0$ satisfy the inequalities in (4.4), they are coupled upper and lower solutions of (4.2). This implies that there exist nonnegative quasisolutions $\bar{u}_s$, $\underline{u}_s$ which satisfy the relation

$$\left.\begin{aligned} -L\bar{u}_s &= \bar{u}_s(a - b\bar{u}_s - c\underline{u}_s) \\ -L\underline{u}_s &= \underline{u}_s(a - b\underline{u}_s - c\bar{u}_s) \end{aligned}\right. \quad \text{in } \Omega$$

$$B\bar{u}_s = B\underline{u}_s = 0 \qquad \text{on } \partial\Omega \tag{4.6}$$

(cf. [12]). Moreover, by choosing $M \geq \eta(t, x)$ in Q_0, Theorem 2.2 ensures that problem (4.1) has a unique global solution $u^*(t, x)$ and $0 \leq u^*(t, x) \leq M$ for all $t > 0$, $x \in \overline{\Omega}$. In view of Theorem 3.2, $\underline{u}_s(x) \leq u^*(t, x) \leq \bar{u}_s(x)$ as $t \to \infty$.

To investigate the asymptotic limit of $u^*(t, x)$ as $t \to \infty$ we first consider the case $\lambda_0 \geq 1$. Since $\bar{u}_s$ and $\underline{u}_s$ are the limits of the sequences $\{\bar{u}^{(m)}\}$, $\{\underline{u}^{(m)}\}$ governed by (3.4) with $\bar{u}^{(0)} = M$, $\underline{u}^{(0)} = 0$, and $f(x, u, J * u)$ is given by (4.3) we see that $\underline{u}^{(m)} = 0$ for all m and $\bar{u}_s$ is a solution of the scalar boundary-value problem

$$-LU_s = U_s(a - bU_s) \text{ in } \Omega, \qquad BU_s = 0 \text{ on } \partial\Omega. \tag{4.7}$$

But $U_s = 0$ is the only solution of (4.7) when $\lambda_0 \geq 1$ we see that $\bar{u}_s = \underline{u}_s = 0$. It follows from Theorem 3.2 that $u^*(t, x) \to 0$ as $t \to \infty$. The arbitrariness of M implies that the trivial solution is globally asymptotically stable (with respect to nonnegative initial perturbations). On the other hand, if $\lambda_0 < 1$ then problem (4.2) has a unique positive solution $u_s^*(x)$. Unlike the problem without time delay it is not clear whether the solution $u^*(t,x)$ converges to $u^*(x)$ without additional conditions (cf. [12, 14, 15, 18]). We show that if $b > c$ in $\overline{\Omega}$ then $u^*(t,x) \to u^*(x)$ as $t \to \infty$ for a class of initial functions $\eta(t,x)$ when the time delay is infinite and for every $\eta(t,x) \geq 0$ with $\eta(0,x) \neq 0$ when the time delay is finite. To show this, we use some results from [12] which state that for any positive constants ρ, δ with $1 \leq \rho < b/c$ and δ sufficiently small the pair $\tilde{u}_s = \rho U_s$ and $\hat{u}_s = \delta U_s$ are coupled upper and lower solutions of (4.2), and the corresponding quasisolutions $\overline{u}_s$, $\underline{u}_s$ are equal and coincide with the positive solution $\overline{u}_s^*$, where U_s is the unique positive solution of (4.7). In view of Theorem 3.2 these results imply that the solution $u^*(t,x)$ of (4.1) with $\eta \in \langle \delta U_s, \rho U_s \rangle$ converges to $\mathbf{u}_s^*(x)$ as $t \to \infty$. Since by the comparison principle for parabolic boundary-value problems without time delay, $u^*(t,x) > 0$ in Ω for some $t_1 > 0$ the arbitrary smallness of $\delta > 0$ implies that $u^*(t,x) \to u_s^*$ for any nontrivial nonnegative $\eta \leq \rho U_s$. Furthermore, if the delay is finite and $I = [-r, 0]$ (continuous or discrete) then the argument in [12] shows that for any nonnegative $\eta(t,x)$ with $\eta(0,x) \not\equiv 0$ there exists a finite $t^* > 0$ such that $\delta U_s \leq u^*(t,x) \leq \rho U_s$ for $t^* - r \leq t \leq t^*$, $x \in \overline{\Omega}$. As a consequence of Theorem 3.3 and $\overline{u}_s = \underline{u}_s = u_s^*$ the solution $u^*(t,x)$ converges to $u_s^*(x)$ as $t \to \infty$. To summarize the above conclusions we have the following

THEOREM 4.1. *Let λ_0 be the smallest eigenvalue of* (4.5) *and let $u^*(t,x)$ be the solution of* (4.1) *with $\eta(t,x) \geq 0$ and $\eta(0,x) \not\equiv 0$. Then as $t \to \infty$, $u^*(t,x) \to 0$ if $\lambda_0 \geq 1$, and $u^*(t,x) \to u_s^*(x)$ if $\lambda_0 < 1$, $b > c$ and $\eta(t,x) \leq \rho U_s$, where $1 \leq \rho < b/c$ and u_s^* and U_s are the respective positive solutions of* (4.2) *and* (4.7). *Moreover, for finite continuous or discrete time delay the convergence of $u^*(t,x)$ to $u_s^*(x)$ holds for every $\eta(t,x) \geq 0$ with $\eta(0,x) \not\equiv 0$.*

(B) *A Volterra–Lotka Cooperative Model*

We next consider a diffusive Volterra–Lotka cooperative model between two species u, v in the form

$$\begin{aligned}
&u_t - D_1 \nabla^2 u = u(a_1 + b_1 u + c_1 J_2 * v) && \\
&v_t - D_2 \nabla^2 v = v(a_2 + b_2 J_1 * u - c_2 v) && (t > 0,\ x \in \Omega) \\
&Bu = 0, \qquad Bv = 0 && (t > 0,\ x \in \partial\Omega) \\
&u(t,x) = \eta_1(t,x), \qquad v(t,x) = \eta_2(t,x) && (t \in I_i,\ x \in \Omega),
\end{aligned} \tag{4.8}$$

where D_i, a_i, b_i, and c_i, $i = 1, 2$, are positive constants, the boundary operator B is the same as that in (4.1), and $J_1 * u$ and $J_2 * v$ are given in the form (1.4) with either discrete or finite continuous delays. The corresponding steady-state problem of (4.8) is given by

$$\begin{aligned} -D_1 \nabla^2 u &= u(a_1 - b_1 u + c_1 v) \text{ in } \Omega, \qquad Bu = 0 \text{ on } \partial\Omega \\ -D_2 \nabla^2 v &= v(a_2 + b_2 u - c_2 v) \text{ in } \Omega, \qquad Bv = 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.9}$$

The dynamics of the system (4.8) in relation to the steady-state solutions of (4.9) for the case without time delay (i.e., $r_1 = r_2 = 0$) has been treated in [9]. It is known from [9] (see also [1, 4]) that problem (4.9) has only the trivial solution $(0,0)$ if $a_i \le D_i \lambda_0$ and it has at least the trivial and semitrivial solutions $(0,0)$, $(U_s, 0)$, $(0, V_s)$ if $a_i > D_i \lambda_0$, $i = 1,2$, where λ_0 is the smallest eigenvalue of (4.5) with $L = \nabla^2$ and $a(x) \equiv 1$, and U_s and V_s are the respective positive solutions of the scalar boundary-value problems

$$\begin{aligned} -D_1 \nabla^2 U_s &= U_s(a_1 - b_1 U_s) \text{ in } \Omega, \qquad B[U_s] = 0 \text{ on } \partial\Omega \\ -D_2 \nabla^2 V_s &= V_s(a_2 - c_2 V_s) \text{ in } \Omega, \qquad B[V_s] = 0 \text{ on } \partial\Omega. \end{aligned} \tag{4.10}$$

Moreover, for the case $a_i > D_i \lambda_0$, problem (4.9) has a positive maximal solution $\bar{\mathbf{u}}_s \equiv (\bar{u}_s, \bar{v}_s)$ and a positive minimal solution $\underline{\mathbf{u}}_s \equiv (\underline{u}_s, \underline{v}_s)$ if $b_2 c_1 < b_1 c_2$, and it has no positive solution if $b_2 c_1 > b_1 c_2$. Concerning the dynamics of the time-dependent system (4.8) without delays we have the following results from [9].

THEOREM B. *Let (u, v) be the (local) solution of (4.8) corresponding to $r_i = 0$, $\eta_i \equiv \eta_i(x)$ with $\eta_i(x) \not\equiv 0$, $i = 1,2$. If $a_i < D_i \lambda_0$ and $b_2 c_1 \le b_1 c_2$, $i = 1,2$, then (u, v) exists globally and converges to $(0,0)$ as $t \to \infty$. On the other hand, if $a_i > D_i \lambda_0$, $i = 1,2$, then for $(\eta_1, \eta_2) \le (U_s, V_s)$ the solution (u,v) exists globally and converges to $(\underline{u}_s, \underline{v}_s)$ if $b_2 c_1 < b_1 c_2$, and it blows-up in finite-time if $b_2 c_1 > b_1 c_2$. Moreover the finite time blowing-up property of (u, v) holds for every nontrivial $(\eta_1, \eta_2) \ge (0,0)$ if $b_2 c_1 > b_1 c_2$, and for suitably large (η_1, η_2) if $b_2 c_1 < b_1 c_2$.*

For the problem (4.8) with delay we apply Theorems 2.2 and 3.2 to investigate any possible effect of the delay on the asymptotic behavior of the solution. We first consider the case $a_i < D_i \lambda_0$, $i = 1,2$. Since in this case problem (4.9) has only the trivial solution $(0,0)$, the asymptotic behavior of the solution can be determined by a suitable pair of upper and lower solutions. Here the reaction function $\mathbf{f} \equiv (f_1, f_2)$ in (4.8) is quasi-monotone nondecreasing for all $u \ge 0$, $v \ge 0$, and therefore the requirements of an upper solution of (4.8), denoted by $(\tilde{u}, \tilde{v})$, become

$$\begin{aligned}
&\tilde{u}_t - D_1 \nabla^2 \tilde{u} \ge \tilde{u}\left(a_1 - b_1 \tilde{u} + c_1 J_2 * \tilde{v}\right) \\
&\tilde{v}_t - D_2 \nabla^2 \tilde{v} \ge \tilde{v}\left(a_2 + b_2 J_1 * \tilde{u} - c_2 \tilde{v}\right) \\
&B\tilde{u} \ge 0, \qquad B\tilde{v} \ge 0 \\
&\tilde{u}(t,x) \ge \eta_1(t,x), \qquad \tilde{v}(t,x) \ge \eta_2(t,x).
\end{aligned} \tag{4.11}$$

Similarly, a lower solution $(\hat{u}, \hat{v})$ is required to satisfy the inequalities in (4.11) in reversed order. It is obvious from $(\eta_1, \eta_2) \ge (0,0)$ that $(\hat{u}, \hat{v}) = (0,0)$ is a lower solution. We seek an upper solution in the form $(\tilde{u}, \tilde{v}) = (\rho_1 e^{-\alpha t}\phi,\ \rho_2 e^{-\alpha t}\phi)$, where $\phi \equiv \phi(x)$ is the (normalized) positive eigenfunction of (4.5) corresponding to λ_0, and ρ_1, ρ_2 and α are some positive constants to be chosen. By (4.5), $(\tilde{u}, \tilde{v})$ satisfies the inequalities in (4.11) if $\rho_i e^{-\alpha t}\phi \ge \eta_i(t,x)$ in $Q_0^{(i)}$ and

$$\begin{aligned}
-\alpha + D_1 \lambda_0 &\ge a_1 - b_1 \rho_1 e^{-\alpha t}\phi + c_1 J_2 * \left(\rho_2 e^{-\alpha t}\phi\right) \\
-\alpha + D_2 \lambda_0 &\ge a_2 + b_2 J_1 * \left(\rho_1 e^{-\alpha t}\phi\right) - c_2 \rho_2 e^{-\alpha t}\phi.
\end{aligned} \tag{4.12}$$

Since by (1.4) and (1.7),

$$J_i * \left(\rho_i e^{-\alpha t}\phi\right) = \int_0^{r_i} J_i(s,x)\, \rho_i e^{-\alpha(t-s)}\phi\, ds \le \rho_i e^{-\alpha(t-r_i)}\phi$$

for finite continuous delay, and

$$J_i * \left(\rho_i e^{-\alpha t}\phi\right) = \rho_i e^{-\alpha(t-r_i)}\phi$$

for discrete delay, where $i = 1,2$, relation (4.12) holds if

$$\begin{aligned}
D_1 \lambda_0 - a_1 - \alpha &\ge e^{-\alpha t}\phi\left(c_1 \rho_2 e^{\alpha r_2} - b_1 \rho_1\right) \\
D_2 \lambda_0 - a_2 - \alpha &\ge e^{-\alpha t}\phi\left(b_2 \rho_1 e^{\alpha r_1} - c_2 \rho_2\right).
\end{aligned}$$

In view of $a_i < D_l \lambda_0$ there exists a sufficiently small $\alpha > 0$ such that both inequalities are satisfied whenever $c_1 \rho_2 \le b_1 \rho_1$ and $b_2 \rho_1 \le c_2 \rho_2$, or equivalently, $c_1/b_1 \le \rho_1/\rho_2 \le c_2/b_2$. Hence if $b_2 c_1 \le b_1 c_2$ then there exist arbitrarily large ρ_1, ρ_2 and a small $\alpha > 0$ such that $(\tilde{u}, \tilde{v})$ is a positive upper solution. By Theorem 2.2, a unique global solution (u, v) to (4.8) exists and satisfies the relation

$$(0,0) \le (u,v) \le \left(\rho_1 e^{-\alpha t}\phi, \rho_2 e^{-\alpha t}\phi\right)$$

whenever it holds in Q_0. It follows from the arbitrariness of ρ_1, ρ_2 that for any nonnegative (η_1, η_2) the corresponding solution (u, v) converges to $(0,0)$ as $t \to \infty$.

We next consider the case $a_i > D_i\lambda_0$, $i = 1, 2$. It has been shown in [9] that if $b_2c_1 < b_1c_2$ then for any positive constants ρ and δ_i with $\rho \geq \max\{U_s/\sigma_1, V_s/\sigma_2\}$ and δ_i sufficiently small, where

$$\sigma_1 = (a_1c_2 + a_2c_1)/(b_1c_2 - b_2c_1),$$

$$\sigma_2 = (a_1b_2 + a_2b_1)/(b_1c_2 - b_2c_1)$$

the pair $\tilde{\mathbf{u}}_s \equiv (\rho\sigma_1, \rho\sigma_2)$ and $\hat{\mathbf{u}}_s \equiv (\delta_1\phi, \delta_2\phi)$ are ordered upper and lower solutions of (4.9). This implies that problem (4.9) has a positive maximal solution $\bar{\mathbf{u}}_s \equiv (\bar{u}_s, \bar{v}_s)$ and a positive minimal solution $\underline{\mathbf{u}}_s = (\underline{u}_s, \underline{v}_s)$ such that

$$(\delta_1\phi, \delta_2\phi) \leq (\underline{u}_s, \underline{v}_s) \leq (\bar{u}_s, \bar{v}_s) \leq (\rho\sigma_1, \rho\sigma_2).$$

Moreover, $(\underline{u}_s, \underline{v}_s)$ is the unique positive solution in the sector $\langle \mathbf{0}, \mathbf{U}_s \rangle$ where $\mathbf{U}_s = (U_s, V_s)$ (cf. [9]). Hence by choosing δ_i sufficiently small and ρ sufficiently large Theorem 3.2 ensures that the sector $\langle \underline{\mathbf{u}}_s, \bar{\mathbf{u}}_s \rangle$ is a global attractor of the time-dependent system (4.8) for all $\eta_i(t, x) \geq 0$ with $\eta_i(0, x) > 0$, $i = 1, 2$. In fact, since by the maximum principle the solution (u, v) is positive in Ω for $t > 0$, the sector $\langle \underline{\mathbf{u}}_s, \bar{\mathbf{u}}_s \rangle$ is a global attractor for all nonnegative η_i with $\eta_i(0, x) \not\equiv 0$. Furthermore, the quasimonotone nondecreasing property of (f_1, f_2), and the uniqueness of the solution $(\underline{u}_s, \underline{v}_s)$ in $\langle \mathbf{0}, \mathbf{U}_s \rangle$ imply that for any $(\eta_1, \eta_2) \in \langle \mathbf{0}, \mathbf{U}_s \rangle$ with $\eta_i(0, x) \not\equiv 0$ the solution (u, v) converges to $(\underline{u}_s, \underline{v}_s)$ as $t \to \infty$.

On the other hand, if $b_2c_1 > b_1c_2$ then problem (4.9) has no positive solution, and for any nonnegative (η_1, η_2) with $\eta_i(0, x) \not\equiv 0$ the solution of (4.8) without time delays (i.e., $r_1 = r_2 = 0$) blows up in finite time. However, for the problem with finite time delay, the values of $J_1 * u$ and $J_2 * v$ are known in $D_r \equiv (0, r] \times \Omega$ and the two equations for u and v in (4.8) are uncoupled in D_r, where $r = \min\{r_1, r_2\}$. It follows that a unique solution (u, v) to (4.8) exists in $[0, r] \times \overline{\Omega}$. This implies that $J_1 * u$ and $J_2 * v$ are known in the domain $(r, 2r] \times \Omega$, and therefore problem (4.8) with the initial function $(u(r, x), v(r, x))$ has a unique solution (u, v) in $[r, 2r] \times \overline{\Omega}$. By the existence-uniqueness result of Theorem 2.2, (u, v) is the unique solution of (4.8) in $[0, 2r] \times \overline{\Omega}$. A continuation of this process shows that problem (4.8) has a unique global solution (u, v) for any nonnegative (η_1, η_2). In particular, if $(\eta_1, \eta_2) = (\hat{u}_s, \hat{v}_s) \equiv (\delta_1\phi, \delta_2\phi)$ then by Theorem 3.1 the corresponding solution $(\underline{u}, \underline{v})$ is monotone nondecreasing in t. This implies that $(\underline{u}, \underline{v})$ must grow unbounded as $t \to \infty$, for if this were not true then (u, v) would converge to a positive solution of (4.9) contrary to the fact that problem (4.9) has no positive solution when $b_2c_1 > b_1c_2$. Since δ_1 and δ_2 can be chosen arbitrarily small and every solution (u, v) of (4.8) satisfies the relation $(u, v) \geq (\underline{u}, \underline{v})$ when $(\eta_1, \eta_2) \geq (\delta_1\phi, \delta_2\phi)$ we see that all positive solutions of (4.8) grows unbounded in $\overline{\Omega}$ as $t \to \infty$. The above results lead to the following conclusion.

THEOREM 4.2. *Given any nonnegative initial function* (η_1, η_2) *with* $\eta_i(0, x) \not\equiv 0$, $i = 1, 2$, *problem* (4.8) *with finite delay* (*continuous or discrete*) *has a unique global solution* (u, v). *This solution converges to* $(0, 0)$ *as* $t \to \infty$ *if* $a_i < D_i \lambda_0$ *and* $b_2 c_1 \le b_1 c_2$, $i = 1, 2$. *When* $a_i > D_i \lambda_0$ *and* $b_2 c_1 < b_1 c_2$, (u, v) *is attracted to the sector* $\langle \underline{\mathbf{u}}_s, \bar{\mathbf{u}}_s \rangle$, *and if* $(\eta_1, \eta_2) \le (U_s, V_s)$ *then* $(u, v) \to (\underline{u}_s, \underline{v}_s)$ *as* $t \to \infty$. *On the other hand, if* $a_i > D_i \lambda_0$ *and* $b_2 c_1 > b_1 c_2$ *then* (u, v) *grows unbounded as* $t \to \infty$.

(C) *The Oregonator in a Chemical Reaction*

Our final application is concerned with a chemical reaction model, called oregonator, which involves a system of three equations with non-quasimonotone reaction function. This model is given in the form

$$
\begin{gathered}
u_t - D_1 \nabla^2 u = u(a_1 - b_1 u - c_1 v) + d_1 J_2 * v \\
v_t - D_2 \nabla^2 v = -a_2 v - b_2 uv + d_2 J_3 * w \\
w_t - D_3 \nabla^2 w = -a_3 w + d_3 J_1 * u \\
B_1 u = 0, \qquad B_2 v = 0, \qquad B_3 w = 0 \\
u(t, x) = \eta_1(t, x), \qquad v(t, x) = \eta_2(t, x), \qquad w(t, x) = \eta_3(t, x)
\end{gathered}
\tag{4.13}
$$

where D_i, a_i, b_i, c_i, and d_i, $i = 1, 2, 3$, are positive constants and $J_i * u_i$ with $(u_1, u_2, u_3) = (u, v, w)$ are given by (1.4). The above model without time delay is a modified version of the well-known Belousov–Zhabotinskii reaction mechanism and has been given considerable attention in the literature (e.g., see [2, 9, 17]). Here the reaction function $\mathbf{f} \equiv (f_1, f_2, f_3)$ is not mixed quasimonotone unless $c_1 = 0$ or $d_1 = 0$. In view of Definition 2.1 upper and lower solutions, denoted by $(\tilde{u}, \tilde{v}, \tilde{w})$ and $(\hat{u}, \hat{v}, \hat{w})$, respectively, are required to satisfy the boundary-initial inequalities in (2.1) (with $h_i = 0$, $i = 1, 2, 3$) and the differential inequalities

$$
\begin{aligned}
&\tilde{u}_1 - D_1 \nabla^2 \tilde{u} \ge \tilde{u}\big(a_1 - b_1 \tilde{u} - c_1 \hat{v}\big) + d_1 J_2 * \tilde{v} \\
&\tilde{v}_t - D_2 \nabla^2 \tilde{v} \ge -a_2 \tilde{v} - b_2 \hat{u}\tilde{v} + d_2 J_3 * \tilde{w} \\
&\tilde{w}_t - D_3 \nabla^2 \tilde{w} \ge -a_3 \tilde{w} + d_3 J_1 * \tilde{u} \\
&\hat{u}_t - D_1 \nabla^2 \hat{u} \le \hat{u}\big(a_1 - b_1 \hat{u} - c_1 \tilde{v}\big) + d_1 J_2 * \hat{v} \\
&\hat{v}_t - D_2 \nabla^2 \hat{v} \le -a_2 \hat{v} - b_2 \tilde{u}\hat{v} + d_2 J_3 * \hat{w} \\
&\hat{w}_t - D_3 \nabla^2 \hat{w} \le -a_3 \hat{w} + d_3 J_1 * \hat{u}.
\end{aligned}
\tag{4.14}
$$

It is easy to see that the constant pair $(\tilde{u}, \tilde{v}, \tilde{w}) = (M_1, M_2, M_3)$ and $(\hat{u}, \hat{v}, \hat{w}) = (0, 0, 0)$ satisfy all the inequalities in (4.14) and the boundary-initial requirements in (2.1) if

$$0 \geq M_1(a_1 - b_1 M_1) + d_1 J_2 * M_2$$
$$0 \geq -a_2 M_2 + d_2 J_3 * M_3$$
$$0 \geq -a_3 M_3 + d_3 J_1 * M_1.$$

Since $J_i * M_i = M_i$, $i = 1, 2, 3$, the above requirements are fulfilled by any positive constants M_i such that

$$M_1 \geq \frac{a_1}{b_1} + \frac{d_1 d_2 d_3}{b_1 a_2 a_3}, \qquad M_2 = \frac{d_2 d_3}{a_2 a_3} M_1, \qquad M_3 = \frac{d_3}{a_3} M_1.$$

By choosing M_1 sufficiently large, if necessary, we conclude that the pair $\mathbf{M} \equiv (M_1, M_2, M_3)$ and $\mathbf{0} \equiv (0, 0, 0)$ are coupled upper and lower solutions of (4.13) for any nonnegative (η_1, η_2, η_3). It is obvious that this pair are also coupled upper and lower solutions of the steady-state problem of (4.13). By Theorem 2.2, problem (4.13) has a unique global solution $\mathbf{u} \equiv (u, v, w)$ such that $\mathbf{0} \leq \mathbf{u} \leq \mathbf{M}$ in $\mathbb{R}^+ \times \overline{\Omega}$. Moreover if $a_1 \leq D_1 \lambda_0$ and $d_1 = 0$, where λ_0 is the smallest eigenvalue of (4.5) with $L = \nabla^2$, $B = B_1$ and $a(x) = 1$, then the steady-state problem of (4.13) has only the trivial solution $\mathbf{0}$ (cf. [9]). Since by using $\overline{\mathbf{u}}^{(0)} = \mathbf{M}$ and $\underline{\mathbf{u}}^{(0)} = \mathbf{0}$ as the coupled initial iterations in (3.4) with $\mathbf{f} \equiv (f_1, f_2, f_3)$ given by the functions in (4.13) (and $h_1 = h_2 = h_3 = 0$) the sequence $\{\underline{\mathbf{u}}_s^{(m)}\}$ consists of the single element $\mathbf{0}$ all for m while the sequence $\{\overline{\mathbf{u}}_s^{(m)}\} \equiv \{\overline{u}^{(m)}, \overline{v}^{(m)}, \overline{w}^{(m)}\}$ is governed by

$$-D_1 \nabla^2 \overline{u}^{(m)} + K_1 \overline{u}^{(m)} = K_1 \overline{u}^{(m-1)} + \overline{u}^{(m-1)}\left(a_1 - b_1 \overline{u}^{(m-1)}\right)$$
$$-D_2 \nabla^2 \overline{v}^{(m)} + K_2 \overline{v}^{(m)} = (K_2 - a_2) \overline{v}^{(m-1)} + d_2 \overline{w}^{(m-1)}$$
$$-D_3 \nabla^2 \overline{w}^{(m)} + K_3 \overline{w}^{(m)} = (K_3 - a_3) \overline{w}^{(m-1)} + d_3 \overline{u}^{(m-1)}$$
$$B_1 \overline{u}^{(m)} = B_2 \overline{v}^{(m)} = B_3 \overline{w}^{(m)} = 0.$$

It follows from $a_1 \leq D_1 \lambda_0$ that $\overline{u}^{(m)} \to 0$ as $m \to \infty$, and therefore $\overline{w}^{(m)} \to 0$ and $\overline{v}^{(m)} \to 0$ as $m \to \infty$. This leads to $\overline{\mathbf{u}}_s = \underline{\mathbf{u}}_s = \mathbf{0}$ where $\overline{\mathbf{u}}_s = \lim \overline{\mathbf{u}}_s^{(m)}$ as $m \to \infty$. Since (f_1, f_2, f_3) is mixed quasimonotone when $d_1 = 0$ we have the following conclusion from Theorem 3.2.

THEOREM 4.3. *Given any* $(\eta_1, \eta_2, \eta_3) \geq (0, 0, 0)$ *there exist positive constants* M_i, $i = 1, 2, 3$, *such that a unique global solution* (u, v, w) *to* (4.13) *exists and satisfies the relation*

$$(0, 0, 0) \leq (u, v, w) \leq (M_1, M_2, M_3) \qquad (t > 0,\ x \in \overline{\Omega}).$$

Moreover, $(u, v, w) \to (0, 0, 0)$ *as* $t \to \infty$ *if* $a_1 \leq D_1 \lambda_0$ *and* $d_1 = 0$.

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